

Chapter Five

Chapter Five

ConceptTests for Section 5.1

1. A drug leaves the body at the rate of 20 ng/min at the start of a 10-minute period, and the rate of decrease continually declines until it is 8 ng/min at the end of the time period. The total decrease in the amount of the drug in the body during this time period might be:

- (a) 150 ng
- (b) 12 ng
- (c) 200 ng/min
- (d) 300 ng
- (e) 15 ng/min

ANSWER:

(a). An upper estimate for the total decrease is $(20 \text{ ng/min}) \cdot (10 \text{ min}) = 200 \text{ ng}$ (nanograms), and a lower estimate is $(8 \text{ ng/min}) \cdot (10 \text{ min}) = 80 \text{ ng}$. The total decrease must be between 80 ng and 200 ng. Notice also that the units of the total change are ng, not the units of the rate of change, ng/min.

2. Figure 5.1 shows the velocities of two cyclists traveling in the same direction. If initially the two cyclists are alongside each other, when does Cyclist 2 overtake Cyclist 1?

- (a) Between 0.75 and 1.25 minutes
- (b) Between 1.25 and 1.75 minutes
- (c) Between 1.75 and 2.25 minutes

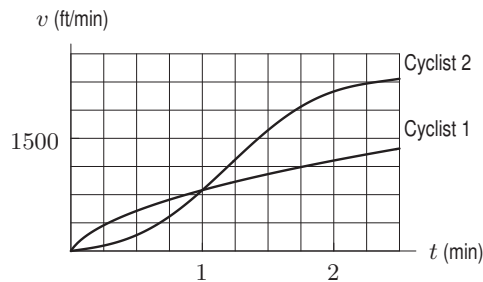


Figure 5.1

ANSWER:

(b). Between 1.25 and 1.75 minutes, because the area under the two curves is about equal at some time during this interval.

COMMENT:

You could ask the students to answer the question if either, or both, of the statements “traveling in the same direction” or “If initially the two cyclists were alongside each other” are removed. You could also use this question to review the concepts of derivative.

Follow-up Question. When the cyclists pass each other, which one is accelerating more?

Answer. Cyclist 2 is accelerating more because between 1.25 and 1.75 minutes the slope of the velocity graph at every point in that interval is steeper than the slope corresponding to Cyclist 1.

3. The table gives a car’s velocity at time t . In each of the four fifteen-minute intervals, the car is either always speeding up or always slowing down. The car’s route is a straight line with four towns on it. Town A is 60 miles from the starting point, town B is 70 miles from the starting point, town C is 73 miles from the starting point, and town D is 80 miles from the starting point. For each town, decide if the car has passed it, has not yet passed it, or may have passed it.

t (minutes)	0	15	30	45	60
v (miles per hour)	60	75	72	78	65

ANSWER:

Calculating an upper and a lower estimate for the distance traveled, we discover that the car traveled at least 67.25 miles and at most 76.5 miles:

$$\text{Lower estimate} = (60 + 72 + 72 + 65)0.25 = 67.25$$

$$\text{Upper estimate} = (75 + 75 + 78 + 78)0.25 = 76.50.$$

Thus, the car definitely passed town A, may have passed towns B and C, and definitely did not pass town D.

COMMENT:

Note that the velocity of the car is not monotone. Therefore the upper and lower estimates for the distance traveled are not equal to the right and left sum estimates.

ConceptTests for Section 5.2

1. Using Figure 5.2, which of the following is the best estimate of $\int_0^9 f(t)dt$?
- (a) 13
 - (b) 30
 - (c) 3
 - (d) 18
 - (e) 9

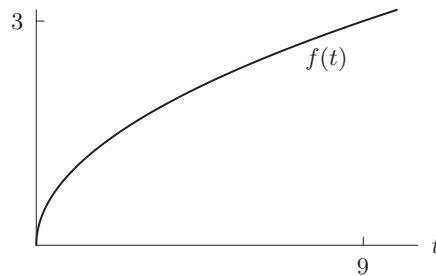


Figure 5.2

ANSWER:

(d). The integral gives the area under the curve from $t = 0$ to $t = 9$. Since the highest point on the curve on this interval is at 3, the area of the entire rectangle is $9 \cdot 3 = 27$. Since the curve fits within this rectangle, the value of the integral must be less than 27. The area under the curve is more than half of the area of the rectangle, so the value of the integral is greater than 13.5. The answer must be (d), 18.

2. An increasing function $f(x)$ has $f(0) = 10$ and $f(5) = 18$. Which of the following is the best estimate of $\int_0^5 f(x)dx$?
- (a) 40
 - (b) 70
 - (c) 100
 - (d) 8
 - (e) 18

ANSWER:

(b). Since the function is increasing and the width of the interval is 5, a lower-estimate for the integral is $10 \cdot 5 = 50$ and an upper-estimate is $18 \cdot 5 = 90$. The value of the integral is between 50 and 90, so the only possible answer is (b) 70. Alternatively, students might sketch a possible graph and estimate the area.

3. Consider the graph in Figure 5.3.
- Give an interval on which the left-hand sum approximation of the area under the curve on that interval is an underestimate.
 - Give an interval on which the left-hand sum approximation of the area under the curve on that interval is an overestimate.

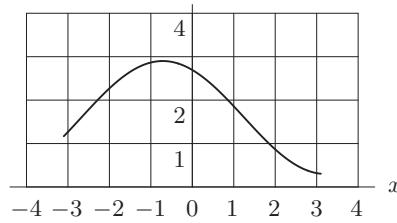


Figure 5.3

ANSWER:

- A possible answer is $[-3, -1]$ because the function is increasing there.
- A possible answer is $[0, 3]$ because the function is decreasing there.

COMMENT:

Follow-up Question. What about right-hand sum?

Answer. On the interval $[0, 3]$ the right-hand sum is an underestimate because the function is decreasing there. On the interval $[-3, -1]$ the right-hand sum is an overestimate because the function is increasing there.

4. Figure 5.4 shows the graph of $y = x^2$. The right-hand sum for eight equal divisions is given by
- $0.5^2 + 1^2 + 1.5^2 + 2^2 + 2.5^2 + 3^2 + 3.5^2 + 4^2$
 - $0.5(0.5) + 1(0.5) + 1.5(0.5) + 2(0.5) + 2.5(0.5) + 3(0.5) + 3.5(0.5) + 4(0.5)$
 - $0.5^2(0.5) + 1^2(0.5) + 1.5^2(0.5) + 2^2(0.5) + 2.5^2(0.5) + 3^2(0.5) + 3.5^2(0.5) + 4^2(0.5)$
 - $0^2(0.5) + 0.5^2(0.5) + 1^2(0.5) + 1.5^2(0.5) + 2^2(0.5) + 2.5^2(0.5) + 3^2(0.5) + 3.5^2(0.5)$

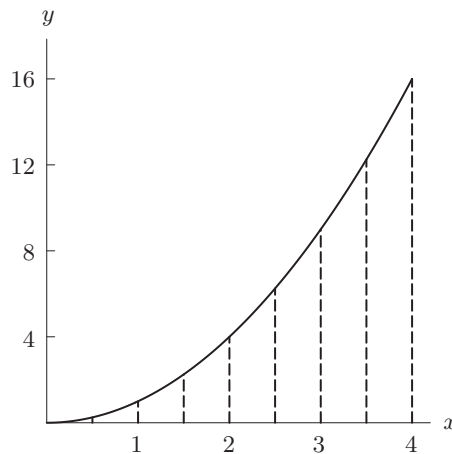


Figure 5.4

ANSWER:

- The width of each rectangle is 0.5, and the x values are squared.

COMMENT:

You could ask the students to give the left-hand sum also. You could also comment on how the right-hand sum gives an overestimate because the function is increasing.

5. For the following question consider continuous curves that are increasing and concave up. As the number of rectangles (of equal width) triples in Figures 5.5 and 5.6, the areas represented by A_1 and A_2 decrease.
- (a) True
 (b) False
 (c) It depends on the graph

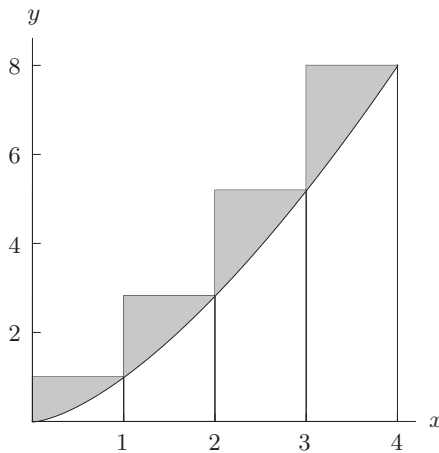


Figure 5.5: $A_1 = \text{Shaded Region}$

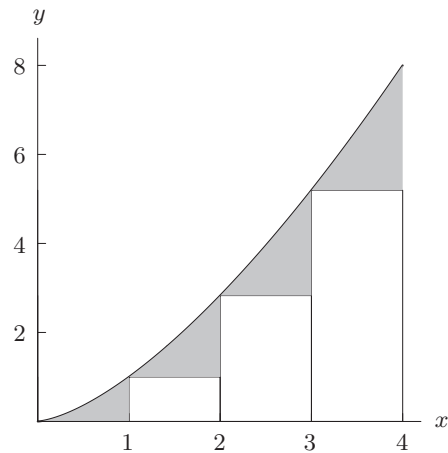


Figure 5.6: $A_2 = \text{Shaded Region}$

ANSWER:

(a). In both graphs, taking more rectangles of equal width will decrease the values of A_1 and A_2 , with the area of the rectangles in both figures becoming closer to the area under the curve.

COMMENT:

You could hand out the graphs to your students and have them subdivide and conjecture. You could ask the students the same question, but without the constraint that the function be increasing and concave up.

6. As the number of rectangles (of equal width) in Figures 5.5 and 5.6 goes to infinity, which of the following statements is true? Consider only continuous curves.
- (a) A_1 goes to zero but A_2 does not. The limit of the lower rectangular area does not equal the limit of the upper rectangular area.
 (b) A_2 goes to zero but A_1 does not. The limit of lower rectangular area does not equal the limit of upper rectangular area.
 (c) A_2 and A_1 don't go to zero. The limits of upper and lower rectangular areas are equal.
 (d) A_1 and A_2 both go to zero. The limits of the upper and lower rectangular areas are equal.
 (e) A_1 and A_2 both go to zero. The limits of the upper and lower rectangular areas are not equal.

ANSWER:

(d)

COMMENT:

Students should realize that all continuous curves have this property. Have them explore this idea by drawing arbitrary continuous curves.

ConceptTests for Section 5.3

1. The graphs of five functions are shown Figure 5.7. The scales on the axes are the same for all five. Which function has the value of its integral $\int_0^{10} f(x)dx$ closest to zero?

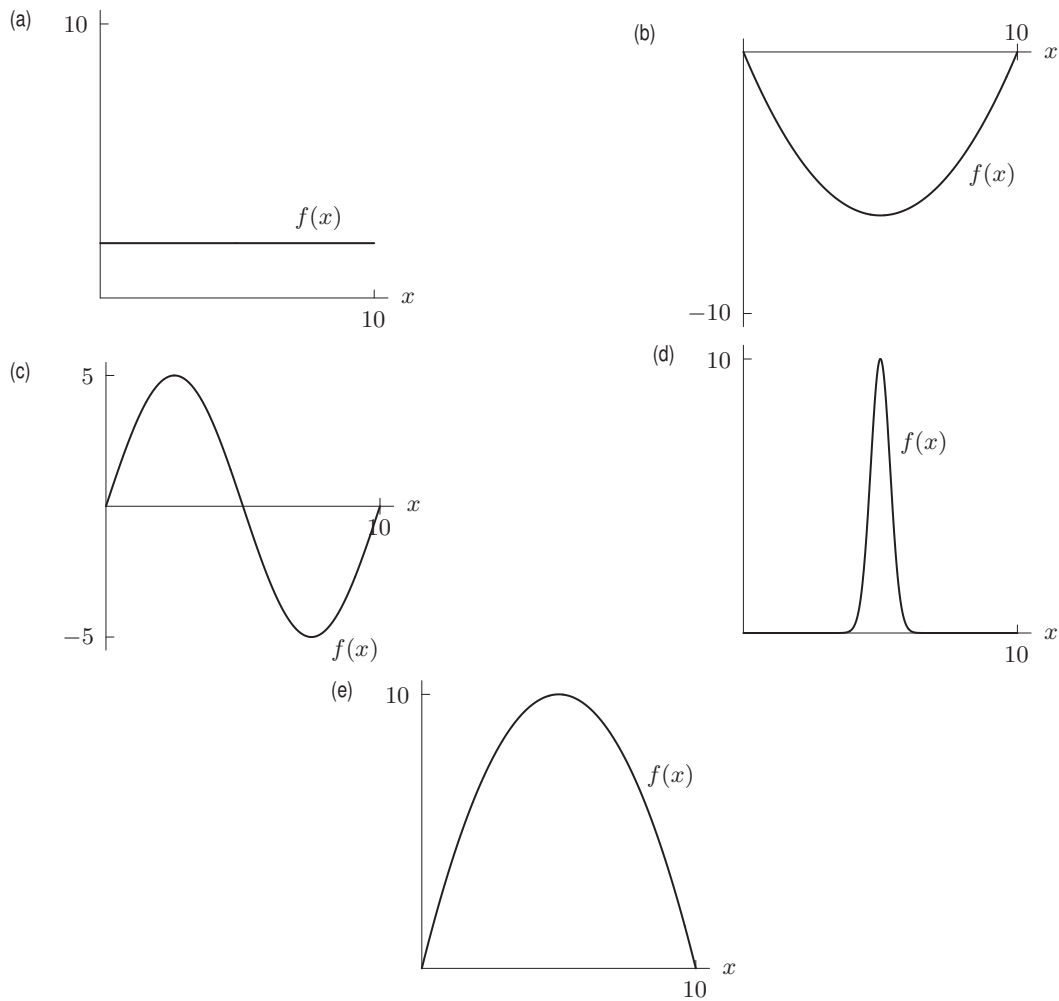
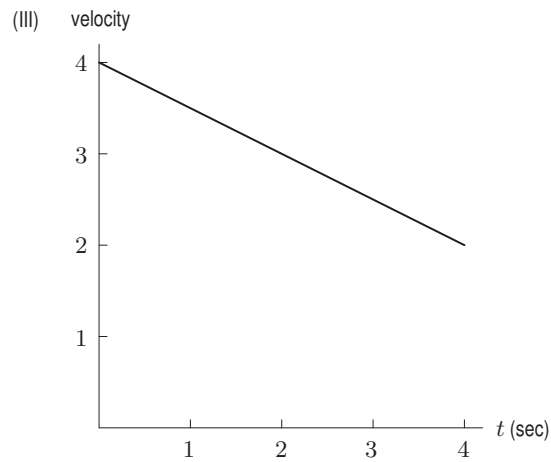
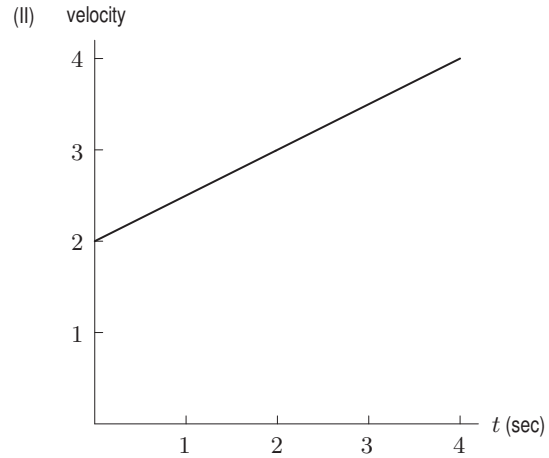
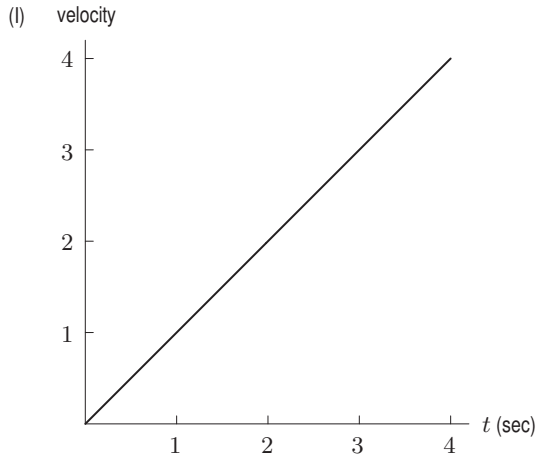


Figure 5.7

ANSWER:

(c). Recall that area above the x -axis counts positively and area below the x -axis counts negatively in an integral. Therefore, the integrals for the functions in (a), (d), and (e) are all positive, and the integral for (b) is negative. In (c), it appears that the area above the axis is approximately equal to the area below the axis, so the integral for the function in (c) will be very close to zero.

2. Graphs (I)–(III) show velocity versus time for three different objects. Order graphs (I)–(III) in terms of the distance traveled in four seconds. (Greatest to least)
- (a) (I), (II), (III)
 (b) (III), (II), (I)
 (c) (II), (III), (I)
 (d) (II) = (III), (I)
 (e) (I) = (II) = (III)



ANSWER:

(d). The distance traveled in this situation is the area under the graph. These areas are (I) $(1/2)(4)(4) = 8$, (II) $(1/2)(4)(4 + 2) = 12$, (III) $(1/2)(4)(4 + 2) = 12$.

COMMENT:

You could mimic this problem using $y = 4 - x$, $y = 4 - x/2$, and $y = 1 + 3x$.

3. Figure 5.8 contains a graph of velocity versus time. Which of the following could be an associated graph of position versus time?

- (a) (I)
(d) (IV)

- (b) (II)
(e) (I), (IV)

- (c) (III)
(f) (II), (III)

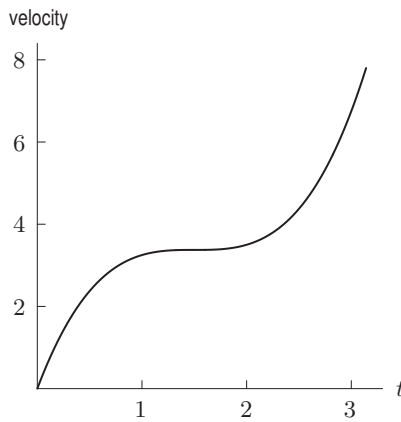
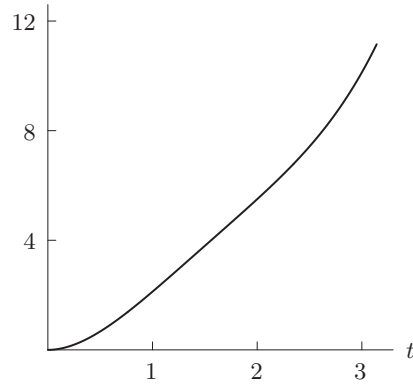
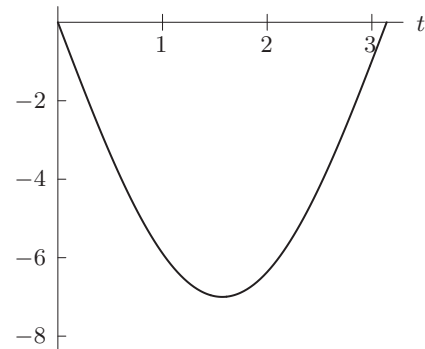


Figure 5.8

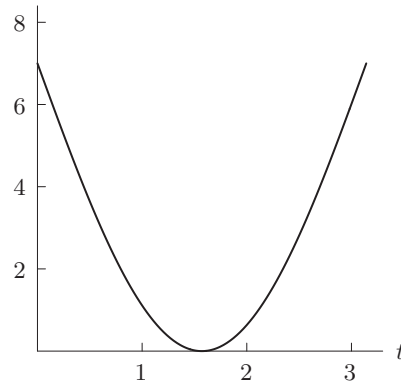
(I) position



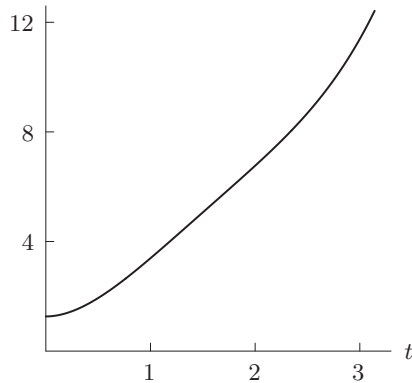
(II) position



(III) position



(IV) position



ANSWER:

(e). Because the velocity is positive for the interval shown, the position versus time graph must be increasing for the interval. Notice graphs (I) and (IV) differ by a vertical shift and both are possible.

COMMENT:

You could start with simpler problems using $y = 1$, then $y = x$, and finally $y = x^2$.

4. Figure 5.9 contains a graph of velocity versus time. Which of (a)–(d) could be an associated graph of position versus time?

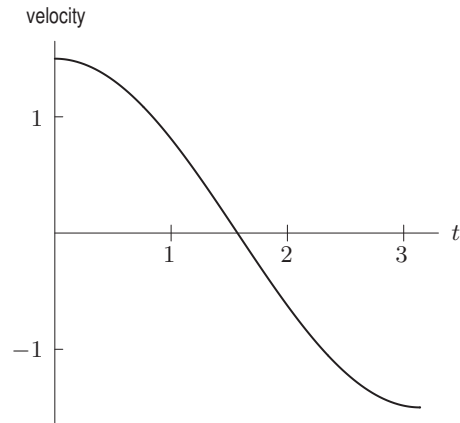
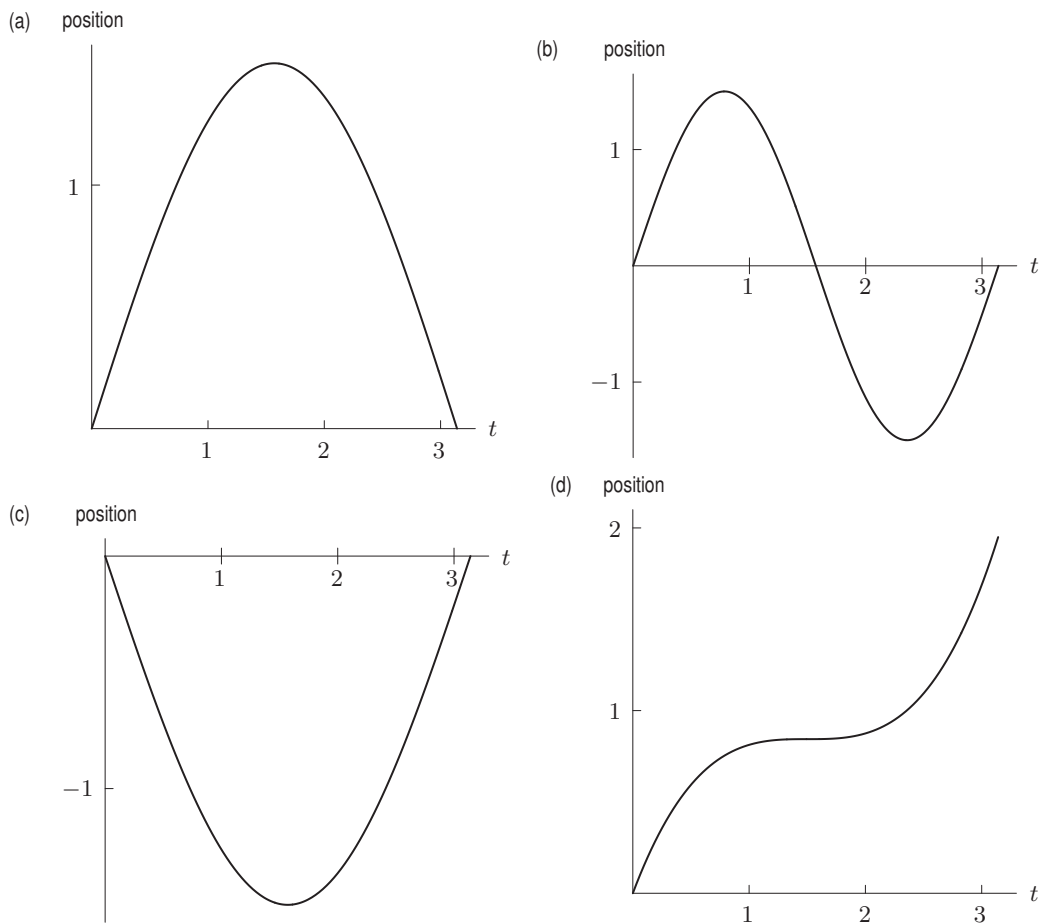


Figure 5.9



ANSWER:

(a). Because the velocity goes from a positive value to a negative value at $x \approx 1.6$, the position will be a maximum there.

COMMENT:

You could elaborate why each of the other choices has properties which exclude it.

5. A bicyclist starts from home and rides back and forth along a straight east/west highway. Her velocity is given in Figure 5.10 (positive velocities indicate travel toward the east, negative toward the west).
- On what time intervals is she stopped?
 - How far from home is she the first time she stops, and in what direction?
 - At what time does she bike past her house?
 - If she maintains her velocity at $t = 11$, how long will it take her to get back home?

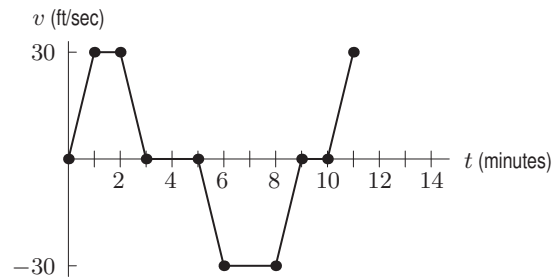


Figure 5.10

ANSWER:

- On $[3, 5]$ and on $[9, 10]$, since $v = 0$ there.
- 3600 feet to the east, since this is the area under the velocity curve between $t = 0$ and $t = 3$.
- At $t = 8$ minutes, since the areas above and below the curve between $t = 0$ and $t = 8$ are equal.
- It will take her 30 seconds longer. By calculating areas, we see that at $t = 11$,

$$\text{Distance from home} = 2 \cdot 30 \cdot 60 - 3 \cdot 30 \cdot 60 + 0.5 \cdot 30 \cdot 60 = -900 \text{ feet.}$$

Thus, at $t = 11$, she is 900 feet west of home. At a velocity of 30 ft/sec eastward, it takes $900/30 = 30$ seconds to get home.

COMMENT:

To get the correct distance we have to convert minutes to seconds.

ConceptTests for Section 5.4

- The units for $f(x)$ are feet per minute (ft/min) and the units of x are feet (ft). What are the units of $\int_0^5 f(x)dx$?
 - ft/min
 - ft
 - (ft)²/min
 - ft/(min)²
 - min
 - 5 ft/min

ANSWER:

(c). The units of the integral are $(f(x)\text{-units}) \cdot (x\text{-units})$ which in this case is $(\text{ft}/\text{min}) \cdot (\text{ft}) = (\text{ft})^2/\text{min}$.

- The function $f(t)$ gives the number of gallons of fuel used per minute by a jet plane t minutes into a flight. The integral $\int_0^{30} f(t)dt$ represents:
 - The average fuel consumption during the first half-hour of the trip.
 - The average fuel consumption during any 30-minute period on the trip.
 - The total fuel consumption during the first 30 minutes of the trip.
 - The total time it takes to use up the first 30 gallons of fuel.
 - The average rate of fuel consumption during the time it takes to use up the first 30 gallons.

ANSWER:

(c). Since $f(t)$ is the rate at which fuel is used, the integral gives the total quantity of fuel used during a particular time interval, here 0 to 30 minutes.

3. In Figure 5.11, the function $f(t)$ gives the rate at which healthy people become sick with the flu, and $g(t)$ is the rate at which they recover. Which of the graphs (a)–(d) could represent the number of people sick with the flu during a 30-day period?

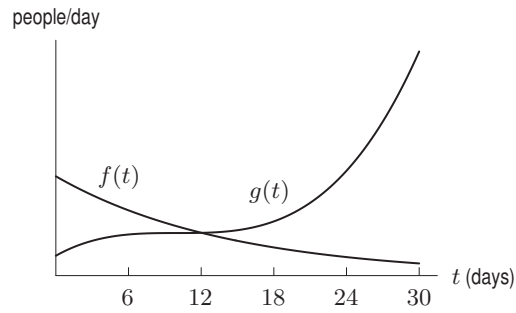
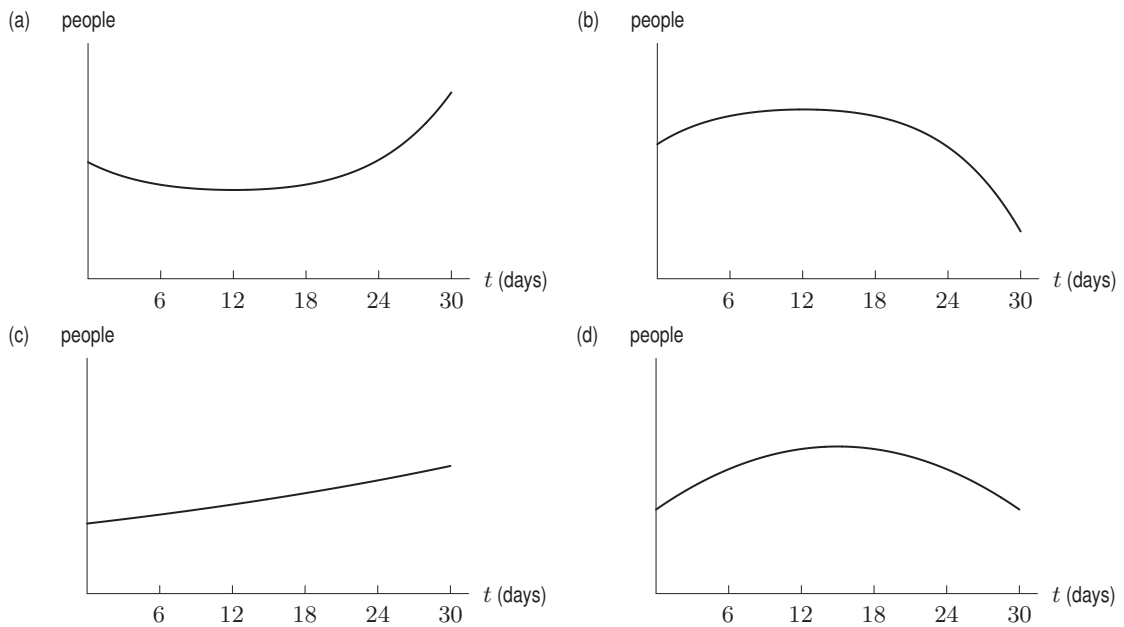


Figure 5.11



ANSWER:

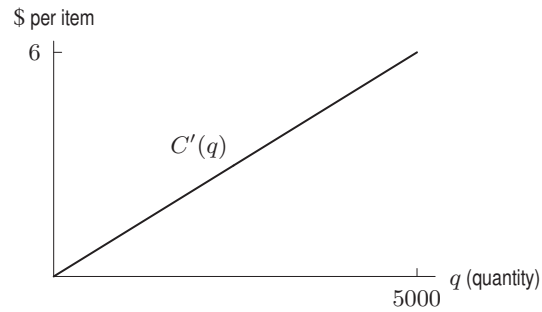
(b). For $t < 12$, we have $f(t) > g(t)$, so the number of sick people is increasing. For $t > 12$, we have $f(t) < g(t)$, so the number of sick people is decreasing. Since the area between the curves is greater for $t > 12$ than the area between the curves for $t < 12$, the decrease is larger than the increase. The answer is (b).

COMMENT:

Ask students what the vertical intercept in graph (b) represents. Ask what they expect to happen with this disease beyond $t = 30$.

ConceptTests for Section 5.5

1. Figure 5.12 shows the marginal cost $C'(q)$. If the fixed cost is \$3000, estimate the total cost of producing 5000 items.
- (a) \$15,000
 - (b) \$45,000
 - (c) \$33,000
 - (d) \$22,500
 - (e) \$18,000

**Figure 5.12**

ANSWER:

(e). The total cost of production is Fixed cost + Variable cost. The total variable cost is the integral of the marginal cost function, which is represented by the area under the curve. Since the region is a triangle, the area is $1/2 \cdot \text{Base} \cdot \text{Height} = (1/2) \cdot 6 \cdot 5000 = 15,000$. Therefore, the total cost is $3000 + 15,000 = \$18,000$.

2. Figure 5.13 contains the graph of $F(x)$, while the graphs in (a)–(d) are those of $F'(x)$. Which shaded region represents $F(b) - F(a)$?

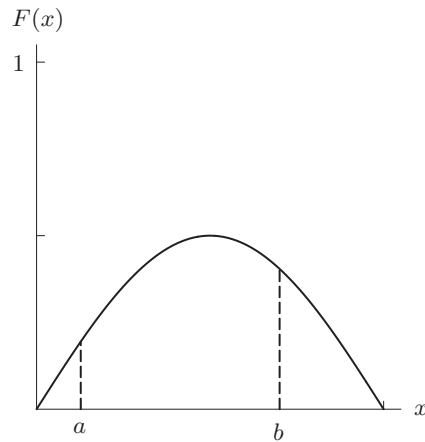
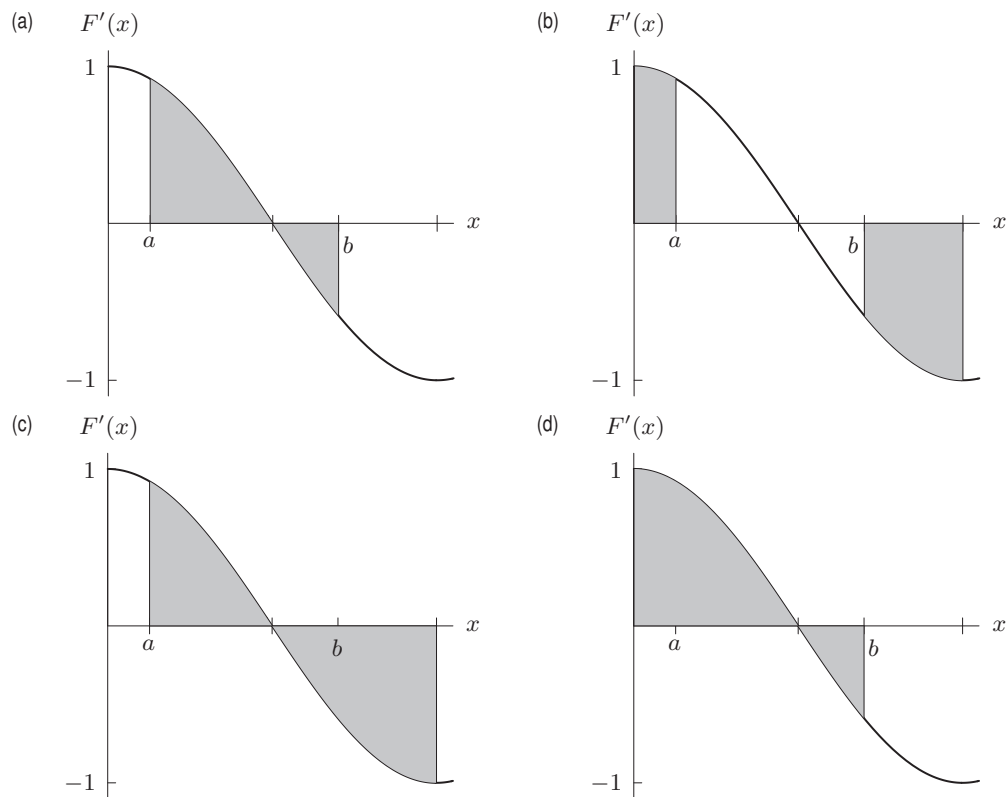


Figure 5.13



ANSWER:

(a). Because $F(b) - F(a) = \int_a^b F'(x) dx$ the left-hand and right-hand limits of the integral must be a and b , respectively.

COMMENT:

Follow-up Question. What feature of the graph of $F'(x)$ tells you that $F(b) > F(a)$?

Answer. More area is shaded above the x -axis than below. Thus, $F(b) - F(a) > 0$ so $F(b) > F(a)$.

ConceptTests for the Second Fundamental Theorem of Calculus

1. If the graph of f is given in Figure 5.14, then which of (a)–(d) is the graph of $\int_{-3}^x f(t) dt$ for $-3 < x < 2$?

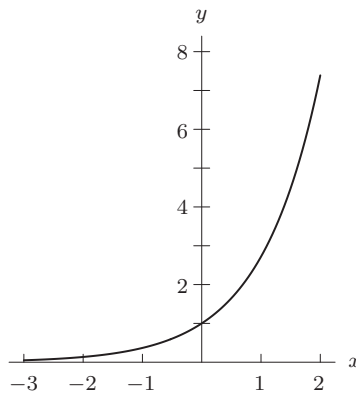
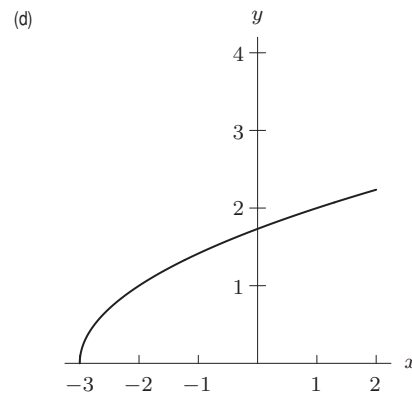
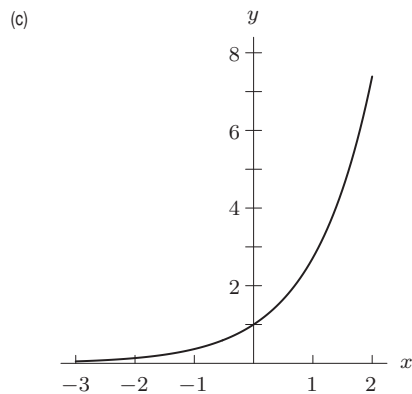
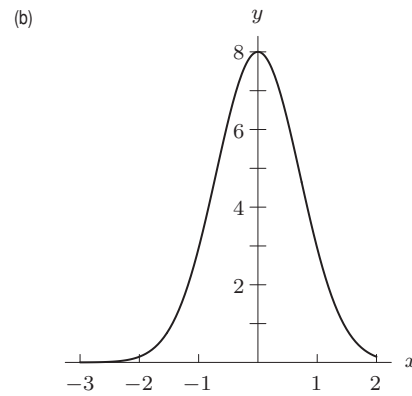
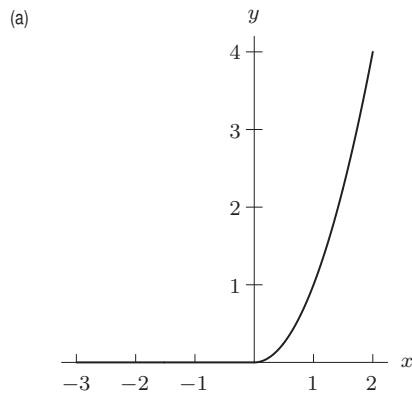


Figure 5.14



ANSWER:

(c). Because $\int_{-3}^{-3} f(t) dt = 0$, the point $(-3, 0)$ is on the graph of $\int_{-3}^x f(t) dt$. Because the graph of f is positive and increasing, $\int_{-3}^x f(t) dt$ will be increasing and concave up.

COMMENT:

Each choice could be examined in detail to show why it is not appropriate.

2. If the graph of f is given in Figure 5.15, then which of (a)–(d) is the graph of $\int_2^x f(t) dt$ for $0 \leq x \leq 4$?

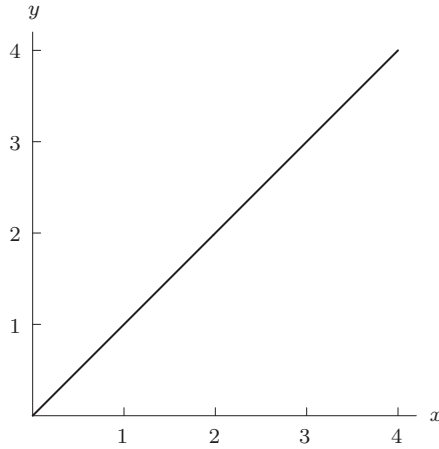
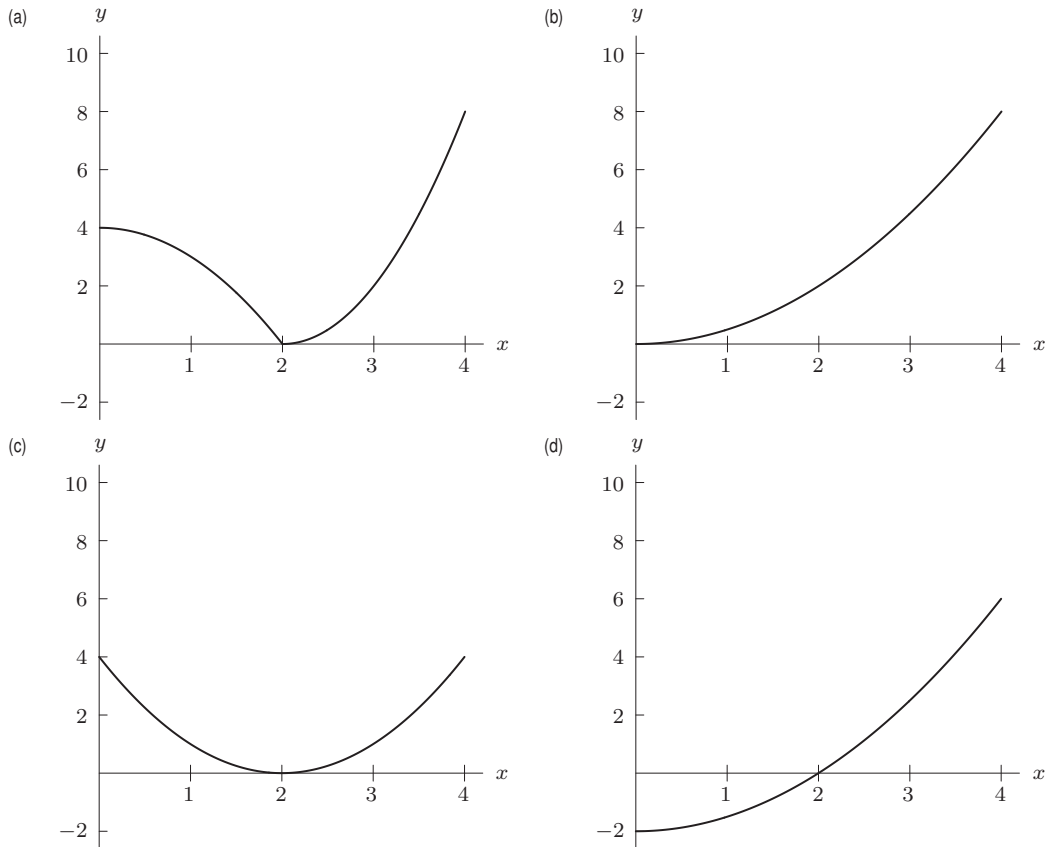


Figure 5.15



ANSWER:

(d). Because $\int_2^2 f(t) dt = 0$, the point $(2, 0)$ is on the graph of $\int_2^x f(t) dt$. Because $\int_2^0 f(t) dt = -\int_0^2 f(t) dt$ and $\int_0^2 f(t) dt$ is positive, then $\int_2^0 f(t) dt$ is negative.

COMMENT:

This example is useful showing the value of interchanging limits.

3. If the graph of f is given in Figure 5.16, then which of (a)–(d) is the graph of $\int_1^x f(t) dt$ for $0 \leq x \leq 3$?

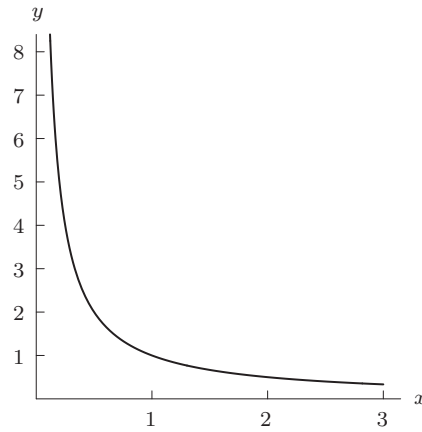
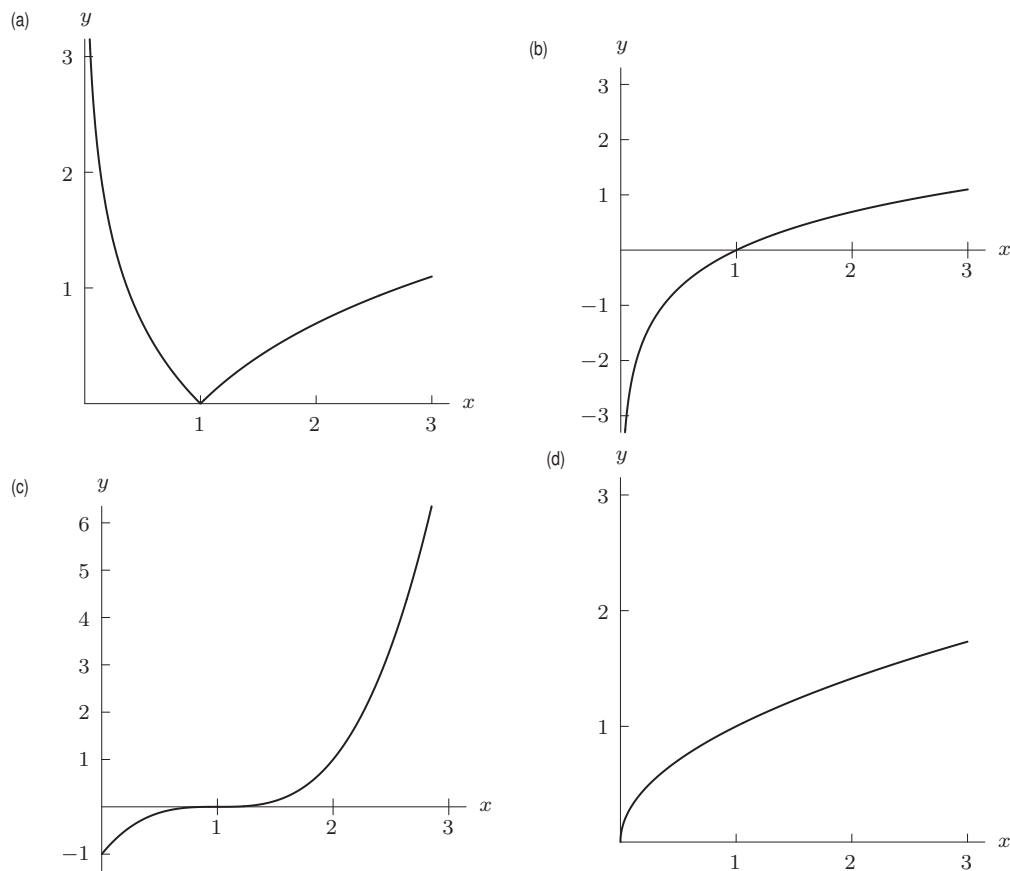


Figure 5.16



ANSWER:

(b). Because $\int_1^1 f(t) dt = 0$, the graph of $\int_1^x f(t) dt$ will contain the point $(1, 0)$. Because $f(x)$ is positive and $\int_1^0 f(t) dt = -\int_0^1 f(t) dt$, then $\int_1^0 f(t) dt < 0$. f is a decreasing function, so $\int_1^x f(t) dt$ will be concave down.

COMMENT:

Have students justify the last statement in the answer.

4. Which of the following graphs (a)–(d) represents the area under the line shown in Figure 5.17 as a function of x ?

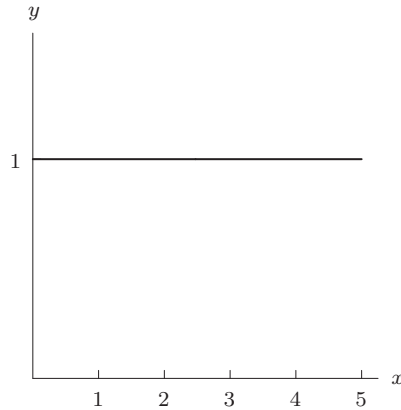
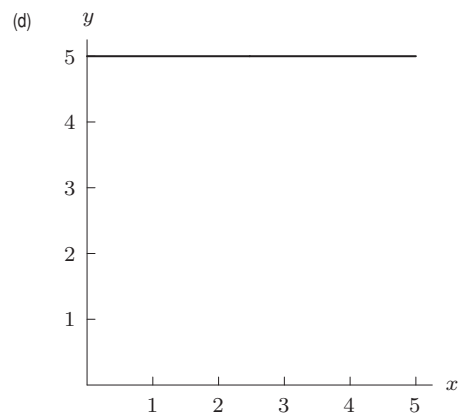
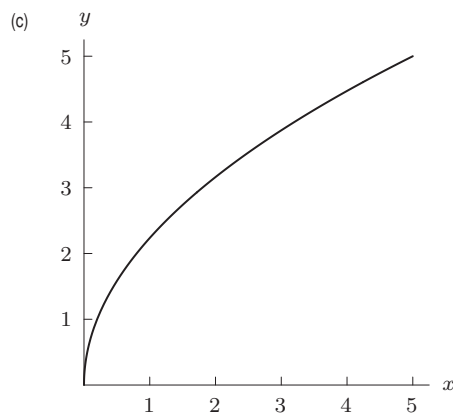
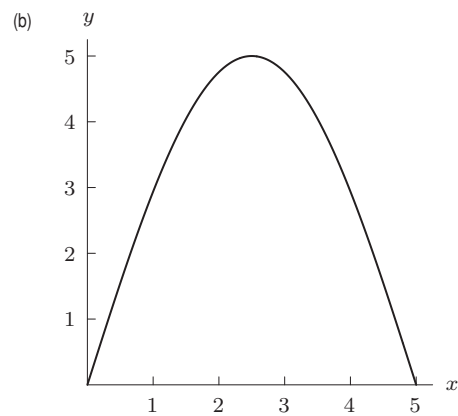
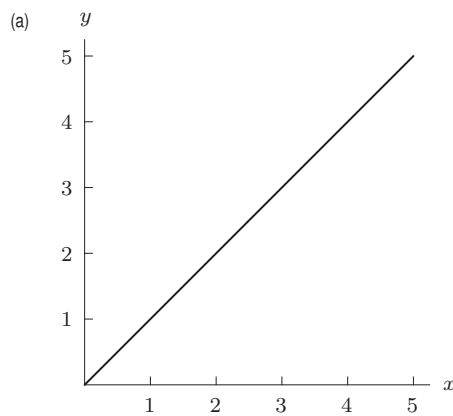


Figure 5.17



ANSWER:

(a). Because the graph shown is that of a positive constant, the area will be directly proportional to the length of the interval.

COMMENT:

You could have your students explain why choice (d) is not the area under any given function from 0 to x .

5. Which of the following graphs (a)–(d) represents the area under the line shown in Figure 5.18 as a function of x ?

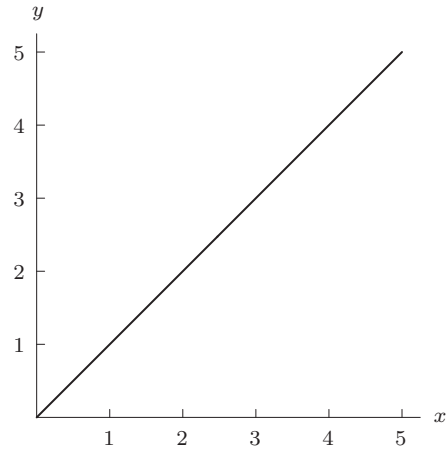
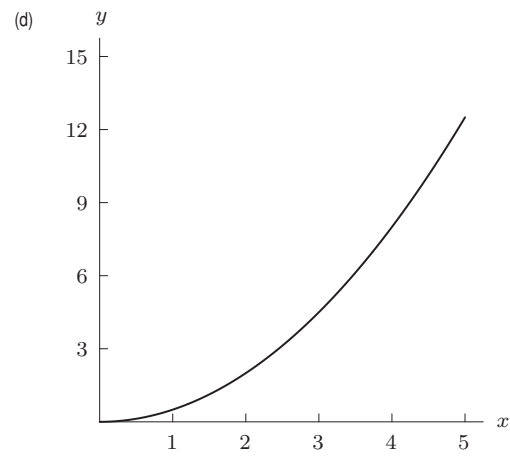
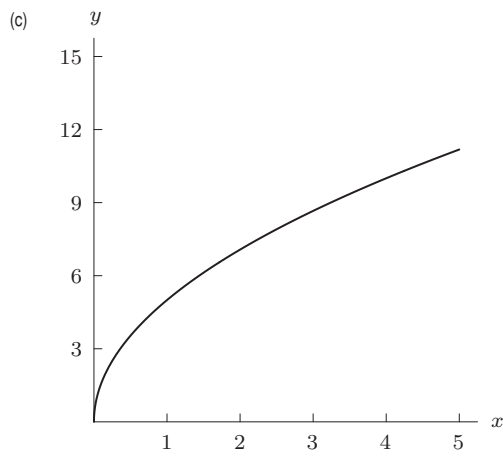
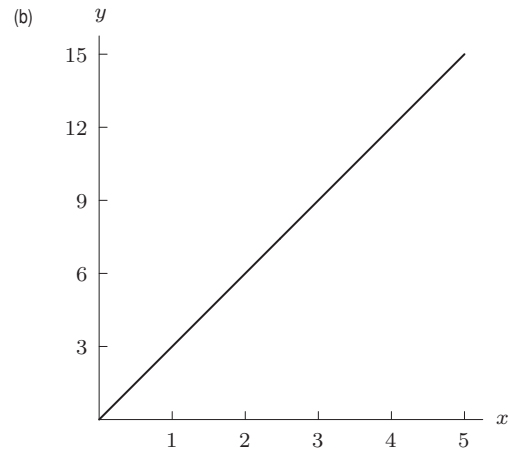
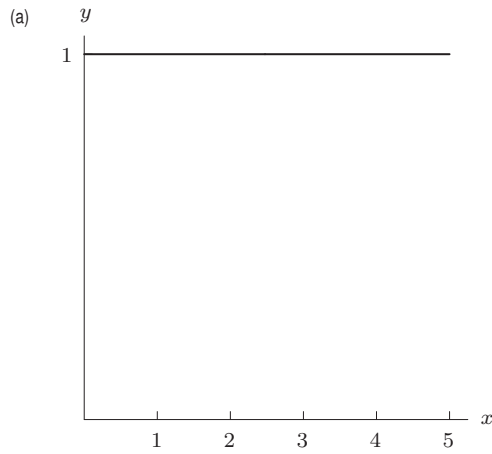


Figure 5.18



ANSWER:

(d). Because the graph in Figure 5.18 is positive and increasing, its antiderivative is increasing and concave up.

COMMENT:

You could ask your students which properties a function must have so that the graph of its area from 0 to x is concave down.

6. Consider the area between the two functions shown in Figure 5.19. Which of the following graphs (a)–(d) represents this area as a function of x ?

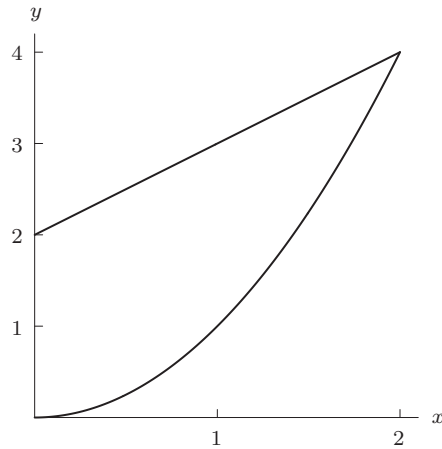
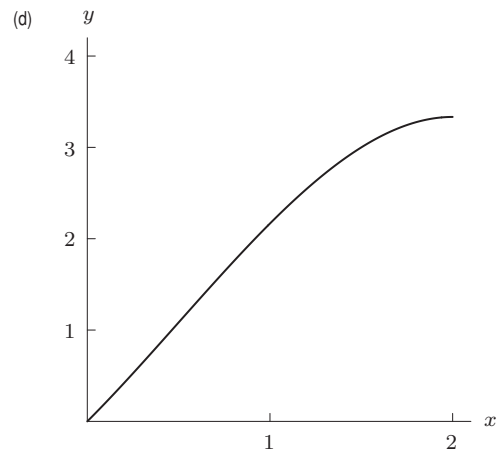
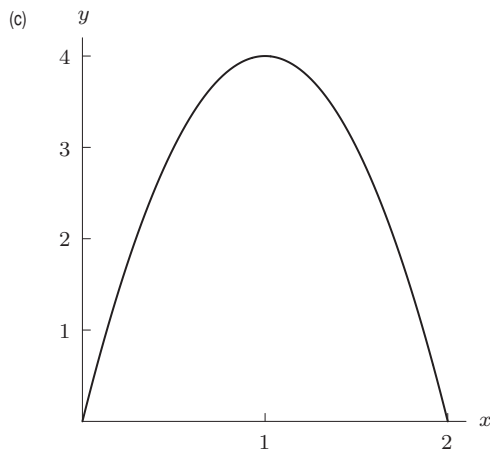
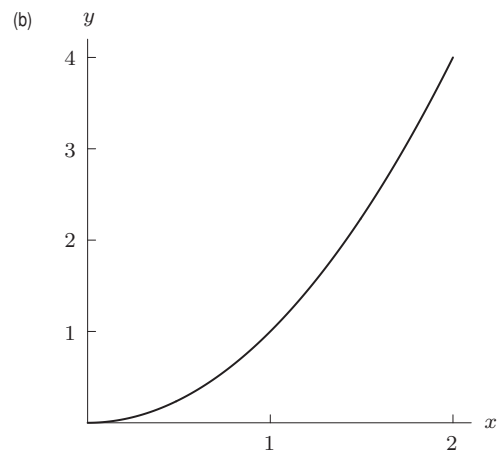
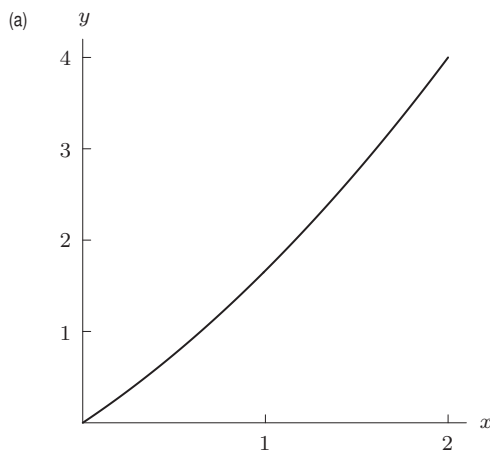


Figure 5.19



ANSWER:

(d) Because the vertical distance between the two curves continually decreases, the graph of the area between these curves will be increasing and concave down.

COMMENT:

Follow-up Question. How can you change the curves in Figure 5.19 so that the graph representing the area between the curves is horizontal for some interval?

Answer. In order for the graph representing the area between the curves to be horizontal on an interval, the area between the graphs is unchanging. Therefore, on that interval the two curves should be equal.

7. Consider the area between the two functions shown in Figure 5.20. Which of the following graphs (a)–(d) represents this area as a function of x ?

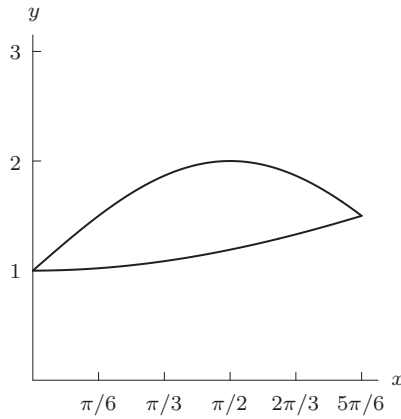
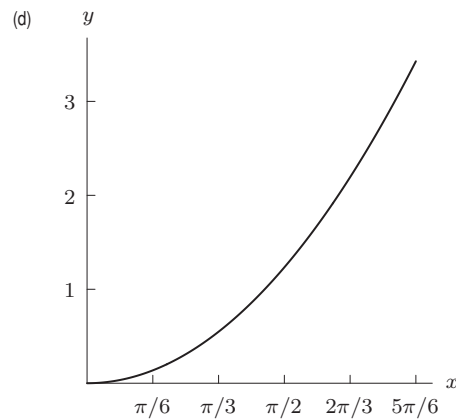
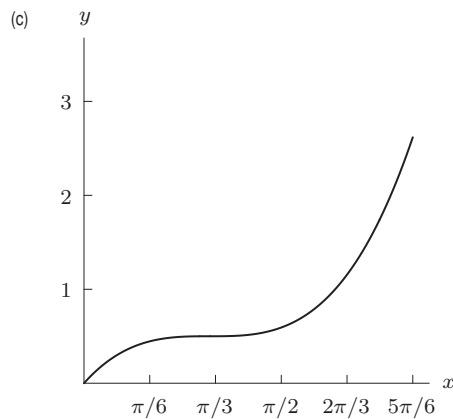
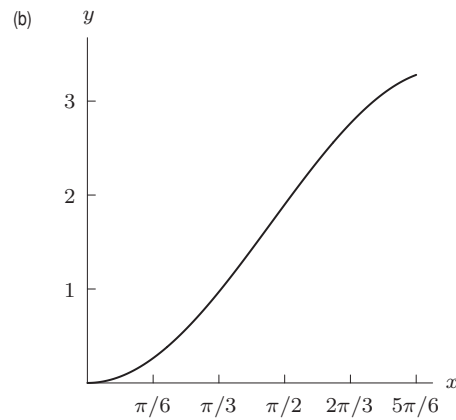
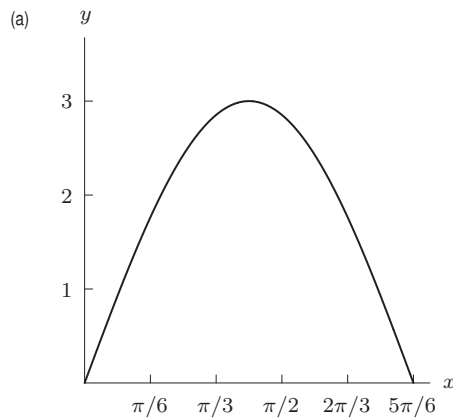


Figure 5.20



ANSWER:

(b). Because the vertical distance between the two curves increases, and then decreases, the graph of the area between them will first be concave up and then concave down.

COMMENT:

Follow-up Question. What situation would cause the area between the two curves to change from concave down to concave up as in choice (c)?

Answer. When $x = 0$ the curves are far apart and move closer to each other until $x \approx \pi/3$. At this point the curves move farther apart.

8. Consider the area between the two functions shown in Figure 5.21. Which of the following graphs (a)–(d) represents this area as a function of x ?

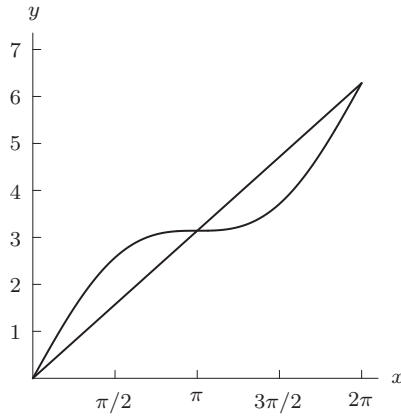
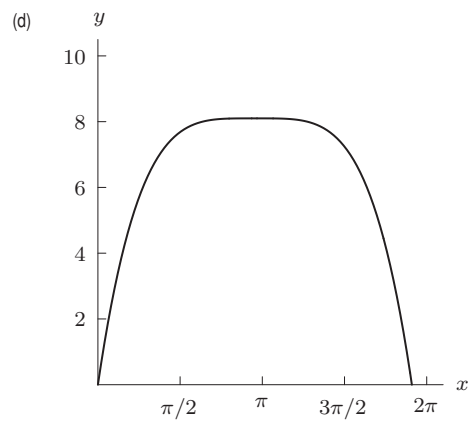
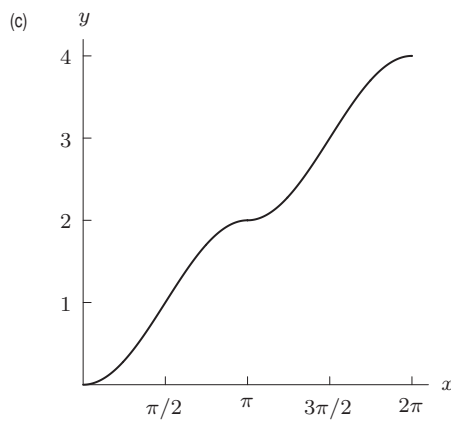
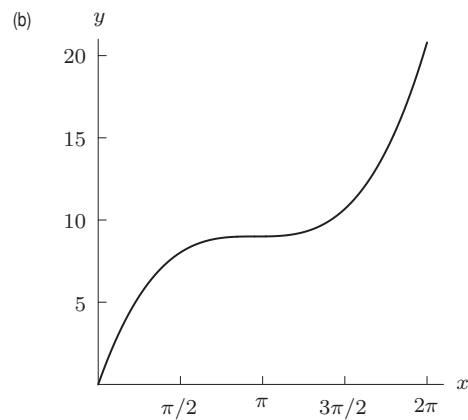
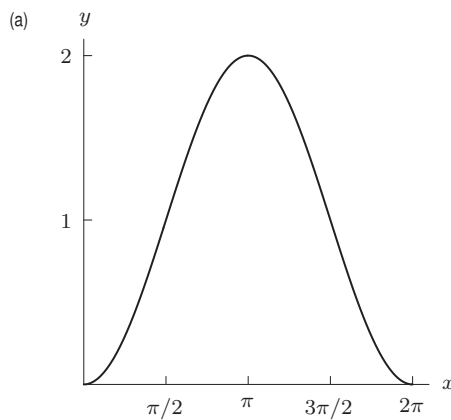


Figure 5.21



ANSWER:

(c). Because the vertical distance between the two curves increases for $0 < x < \pi/2$ and $\pi < x < 3\pi/2$, the graph of the area will be concave up in these intervals. It will be concave down in the other intervals because there the vertical distance between the given curves decreases.

COMMENT:

Follow-up Question. How is the graph of the integral from 0 to x of the difference between these two functions related to the graph in (c)?

Answer. After $x = \pi$, the graph in (c) needs to be reflected across the line $y = 2$ in order to represent the integral of the difference between the curves from 0 to x .