Chapter Six

ConcepTests for Section 6.1

1. Arrange the intervals in the order for which the average value of $\sin x$ over the interval increases from smallest to largest.
   (a) $0 \leq x \leq \pi$
   (b) $\pi/2 \leq x \leq 3\pi/2$
   (c) $\pi \leq x \leq 2\pi$
   (d) $3.14 \leq x \leq 3.15$

   ANSWER:
   (c), (d), (b), (a). On $0 \leq x \leq \pi$, the average value of $\sin x$ is positive; on $\pi/2 \leq x \leq 3\pi/2$, the average value is 0; on $\pi \leq x \leq 2\pi$, the average value is negative. Since $\sin \pi = 0$ and $\pi = 3.14159\ldots$, the sine function changes sign in the interval $3.14 \leq x \leq 3.15$. Since π is closer to the left-end of $3.14 \leq x \leq 3.15$, and because $\sin x$ changes from positive to negative and the graph of $\sin x$ is symmetric about $x = \pi$, the average value of $\sin x$ on this interval is small and negative.
   COMMENT:
   Follow-up Question. What is the average value of $\sin x$ over each interval?
   Answer. (a) The average value is $\frac{1}{\pi}\int_0^{\pi} \sin x\, dx = 0.6366$. (b) The average value is 0. (c) The average value is $\frac{1}{\pi}\int_{\pi}^{2\pi} \sin x\, dx = -0.6366$. (d) The average value is $\frac{1}{0.01}\int_{3.14}^{3.15} \sin x\, dx = -3.407 \times 10^{-3}$.

2. Arrange the intervals in the order for which the average value of $\cos x$ over the interval increases from smallest to largest.
   (a) $0 \leq x \leq \pi$
   (b) $\pi/2 \leq x \leq 3\pi/2$
   (c) $4.71 \leq x \leq 4.72$
   (d) $3.14 \leq x \leq 3.15$

   ANSWER:
   (d), (b), (a), (c). On $0 \leq x \leq \pi$, the average value of $\cos x$ is 0; on $\pi/2 \leq x \leq 3\pi/2$, the average value is negative. Since $3\pi/2$ is closer to the left-end of $4.71 \leq x \leq 4.72$, and because $\cos x$ changes from negative to positive, and the graph of $\cos x$ is symmetric about $x = 3\pi/2$, the average value of $\cos x$ on this interval is small and positive. On $3.14 \leq x \leq 3.15$, the average value is almost -1.
   COMMENT:
   This is an excellent question to explore graphically.

ConcepTests for Section 6.2

1. Which of the following is the source of \$20 consumer surplus?
   (a) A consumer goes to the store to purchase a \$20 item but discovers that the item has sold out.
   (b) A consumer is willing to pay \$100 for an item but manages to buy it for only \$80.
   (c) A store acquires \$300 of stock but is only able to sell \$280 of it.
   (d) A store has \$300 of stock but could have sold \$320 worth if it had been available.

   ANSWER:
   (b) The consumer goes to the store with \$100 and returns with \$20 change and an item he or she values at \$100. The consumer gains \$20 from the trade.
   COMMENT:
   Discuss the fact that different consumers paying the same amount for the same item can gain different amounts from the trade. How is this seen in the demand curve? How is this taken into account in the integral formula for consumer surplus in the text?

2. Which of the following is the source of \$20 producer surplus?
   (a) A producer plans to sell a \$20 item but the customer changes his mind and does not buy it.
   (b) A store has \$300 of stock but could have sold \$320 worth if it had been available.
   (c) A store acquires \$300 of stock but is only able to sell \$280 of it.
   (d) A producer is willing to sell an item for \$100 but manages to sell it for \$120.

   ANSWER:
   (d) The producer trades an item he or she values at \$100 for \$120. The producer gains \$20 from the trade.

COMMENT:

Discuss the fact that different producers selling the same item for the same price can gain different amounts from the trade. How is this reflected in the supply curve? How is this taken into account in the integral formula for producer surplus in the text?

ConcepTests for Section 6.3

1. An income stream has present value \$1000. Which is its value in dollars M years in the future assuming a continuous interest rate of 7% per year?

(a) $M(1000)(1.07)$
(b) $1000(1.07)^M$
(c) $\int_0^M 1000e^{0.07t}\,dt$
(d) $1000e^{0.07M}$

ANSWER:

(d)

COMMENT:

Answer (b) would match a simple (not continuous) interest rate of 7%. Students may be tempted to select (c) because they may associate "income stream" with an integral. They may not realize that the integral they are thinking of has already gone into the computation of the present value which is given in the problem statement.

2. An income is paid continuously for M years at a rate of \$1000 per year. Which is its value in dollars at the end of the M years, assuming a continuous interest rate of 7% per year?

(a) $e^{0.07M}\int_0^M 1000e^{-0.07t}\,dt$
(b) $1000(1.07)^M$
(c) $\int_0^M 1000e^{-0.07t}\,dt$
(d) $1000e^{0.07M}$

ANSWER:

(a) The integral part of the formula, $\int_0^M 1000e^{-0.07t}\,dt$ gives the present value of the income stream at the beginning of the M years. The factor $e^{0.07M}$ converts the present value to a future value.

COMMENT:

Ask the students for situations in which the incorrect three formulas would be appropriate. Formula (b) gives the value M years in the future of \$1000 invested now at 7% simple annual interest. Formula (c) gives the present value of a continuous income stream of \$1000 per year for M years at a continuous interest rate of 7%. And formula (d) gives the value M years in the future of \$1000 invested now at 7% continuous interest.

ConcepTests for Section 6.4

1. A city with initial population 100,000 grows with 2% relative rate of change (annual). What is its population after one year?

(a) $1.02(100{,}000)$
(b) $100{,}000 + 0.02(100{,}000)$
(c) $e^{0.02}100{,}000$
(d) $0.02(100{,}000)$

ANSWER:

(c) Let $P(t)$ be the population of the city after t years. Since the initial population is 100,000, we have

$$\int_0^1 \frac{d}{dt}(\ln P(t))\,dt = \ln P(1) - \ln P(0) = \ln\left(\frac{P(1)}{P(0)}\right) = \ln\left(\frac{P(1)}{100{,}000}\right).$$

Since the relative rate of change is 0.02, we have

$$\frac{d}{dt}(\ln P(t)) = 0.02,$$

so

$$\int_0^1 \frac{d}{dt}(\ln P(t))\,dt = \int_0^1 0.02\,dt = 0.02.$$

Thus

$$\ln\left(\frac{P(1)}{100{,}000}\right) = 0.02,$$

so

$$\frac{P(1)}{100{,}000} = e^{0.02}.$$

The population of the city after one year is $P(1) = e^{0.02}100{,}000$.

COMMENT:

Answers (a) and (b) represent a population growth of 2% during the year and are close approximations of the correct answer. But the relative rate of change is a continuous rate, leading to a slightly larger than 2% population growth in one year. In the continuous model, new population in the first part of the year contributes to additional growth in the later part of the year, leading to the model $P(t) = 100{,}000e^{0.002t}$. You can compare with the difference between annual and continuous interest rates.

2. A city with initial population P_0 grows with 5% relative rate of change (annual). What is its population after t years?
   (a) $(1.05)^t P_0$
   (b) $P_0 + 0.05tP_0$
   (c) $e^{0.05t}P_0$
   (d) $0.05tP_0$

ANSWER:

(c) Let $P(t)$ be the population of the city after t years. Since the initial population is P_0, we have

$$\int_0^b \frac{d}{dt}(\ln P(t))\,dt = \ln P(b) - \ln P(0) = \ln\left(\frac{P(b)}{P_0}\right).$$

Since the relative rate of change is 0.05, we have

$$\frac{d}{dt}(\ln P(t)) = 0.05,$$

so

$$\int_0^b \frac{d}{dt}(\ln P(t))\,dt = \int_0^b 0.05\,dt = 0.05b.$$

Thus

$$\ln\left(\frac{P(b)}{P_0}\right) = 0.05b,$$

so

$$\frac{P(b)}{P_0} = e^{0.05b}.$$

The population of the city after one year is $P(1) = e^{0.05b}P_0$.

COMMENT:

Emphasize that growth with constant relative growth rate is exponential growth. Answers (a) and (c) both give exponential growth, but answer (c) is correct because it corresponds to a continuous relative growth rate.

3. The population of a country grows with 3% relative rate of change. Which of the following can be computed from this information?
   (a) The population after 5 years.
   (b) The change in the population during the first 5 years.
   (c) The percent change in the population during the first 5 years.
   (d) The percent change in the population during any 5 year time period.

ANSWER:

(c) and (d) Actual population, as in (a), and change in population, as in (b), can not be computed without first knowing the initial population of the city.

COMMENT:

Ask the students if the percent change in the population is the same for every 5 year period.