Chapter Nine

ConcepTests for Section 9.1

1. Table 9.1 shows values of $f(x, y)$. Does f appear to be an increasing or decreasing function of x? Of y?

Table 9.1

x \ y	0	5	10	15
0	75	72	68	60
20	80	77	73	68
40	86	82	75	70
60	93	88	82	75

(a) Increasing function of x; Increasing function of y
(b) Increasing function of x; Decreasing function of y
(c) Decreasing function of x; Increasing function of y
(d) Decreasing function of x; Decreasing function of y

ANSWER:
(b). The function f is an increasing function of x (as y is held constant down a column) and a decreasing function of y (as x is held constant across a row.)

2. The total amount, T, of dollars a buyer pays for an automobile is a function of the down payment, P, and the interest rate, r, of the loan on the balance. If the interest rate is held constant, is T an increasing or decreasing function of P?

(a) Increasing
(b) Decreasing
(c) T does not change
(d) There is not enough information to tell

ANSWER:
(b). If the down payment P increases, the total amount that must be paid, T, decreases.
COMMENT:
You could ask to what the special cases of $P = 0$ and $P = T$ correspond.

3. The total amount, T, of dollars a buyer pays for an automobile is a function of the down payment, P, and the interest rate, r, of the loan on the balance. If the down payment is held constant, is T an increasing or decreasing function of r?

(a) Increasing
(b) Decreasing
(c) T does not change
(d) There is not enough information to tell

ANSWER:
(a). If the interest rate r increases, the total amount that must be paid, T, increases.
COMMENT:
You might ask to what the special case $r = 0$ corresponds.

4. The number of songbirds, B, on an island depends on the number of hawks, H, and the number of insects, I. Thinking about the possible relationship between these three populations, decide which of the following depicts it.

(I) B increases as H increases and I remains constant.
(II) B decreases as H increases and I remains constant.
(III) B increases as I increases and H remains constant.
(IV) B decreases as I increases and H remains constant.

(a) I and III
(b) I and IV
(c) II and III
(d) II and IV
(e) None of these

ANSWER:
(c). Under the assumption that hawks eat songbirds, (II) is true. The answer (c) assumes that the songbirds eat the insects. Some songbirds eat only seeds, so in that case, if the insects eat the seeds, the correct answer is (d).
COMMENT:
You might ask for a reason as well.

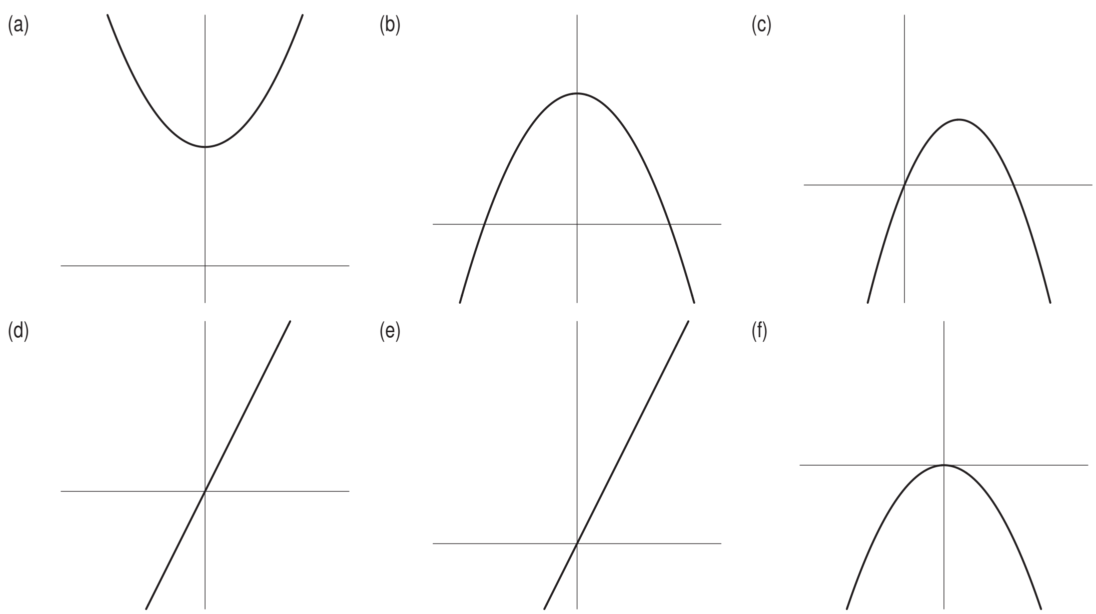
5. For a certain function $z = f(x, y)$, we know that $f(0, 0) = 50$ and that z goes up by 3 units for every unit increase in x and z goes down by 2 units for every unit increase in y. What is $f(2, 5)$?

(a) 51
(b) 46
(c) 1
(d) 55
(e) -4
(f) 16

ANSWER:
(b). At the point $(0, 0)$, the function value is 50. When the x-value goes up by 2, the function value goes up by $3 \cdot 2 = 6$. When the y-value goes up by 5, the function value goes down by $5 \cdot 2 = 10$. We have

$$f(2, 5) = 50 + 6 - 10 = 46.$$

6. Which of the graphs (a)–(f) shows a cross-section of $f(x, y) = 50 - x^2 + 5y$ with y held fixed?



ANSWER:
(b) and (f). When y is fixed, we have $f(x, y) = 50 - x^2 + 5 \cdot C = -x^2 + (50 + 5 \cdot C) = -x^2 + C$ where C is any arbitrary constant. The graph of $y = -x^2 + C$ is an upside-down parabola with its vertex on the y-axis, so the only cross-sections with y fixed are graphs (b) and (f).

ConcepTests for Section 9.2

1. Which of the following terms best describes the origin in the contour diagram in Figure 9.1?

   (a) A mountain pass (b) A dry river bed
   (c) A mountain top (d) A ridge

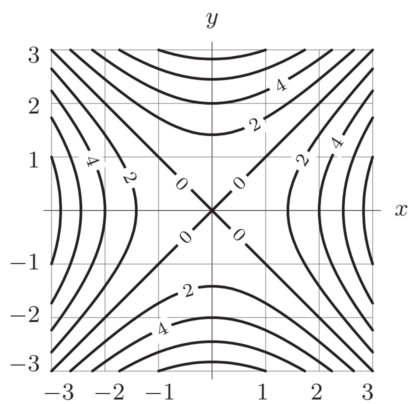


Figure 9.1

ANSWER:
A dry river bed.
COMMENT:
Follow-up Question. Imagine that you are walking on the graph of the function. Describe what you see and how you move.

2. Which of the following is true? If false, find a counterexample.

   (a) The values of contour lines are always 1 unit apart.
   (b) Any contour diagram that consists of parallel lines comes from a plane.
   (c) The contour diagram of any plane consists of parallel lines.
   (d) Contour lines can never cross.
   (e) The closer the contours, the steeper the graph of the function.

ANSWER:
(c) and (e) are true. For (a), Figure 9.32 in the text, a contour diagram of cardiac outputm is a counterexample. For (b), the parabolic cylinder, $z = x^2$, has parallel line contours. For (d), contour lines of the same value can cross, for example, the contour 0 in $z = xy$.
COMMENT:
Follow-up Question. How would you have to change the statements to make them true (if possible)?

3. Draw a contour diagram for the surface $z = 1 - x - y$. Choose a point and determine a path from that point on which

   (a) The altitude of the surface remains constant.
   (b) The altitude increases most quickly.

ANSWER:

(a) To maintain a constant altitude, we move along a contour.
(b) To find the path on which the altitude increases most quickly, we move perpendicular to the level curves.

COMMENT:
Find the equations of these paths in the xy-plane. Choose other paths on the plane and describe them.

4. Figure 9.2 shows contours of a function. You stand on the graph of the function at the point $(-1, -1, 2)$ and start walking parallel to the x-axis. Describe how you move and what you see when you look right and left.

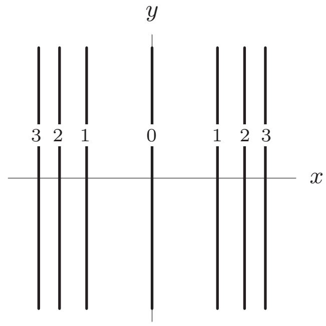


Figure 9.2

ANSWER:

You start to the left of the y-axis on the contour labeled 2. At first you go downhill, and the slope is steep. Then it flattens out. You reach a lowest point right on the y axis and start going uphill; the slope is increasing as you go. To your left and right the terrain is doing the exact same thing; you are walking across a valley that extends to your right and left.

COMMENT:

Follow-up Question. What happens if you walk in a different direction?

ConcepTests for Section 9.3

Problems 1–2 concern the contour diagram for a function $f(x, y)$ in Figure 9.3.

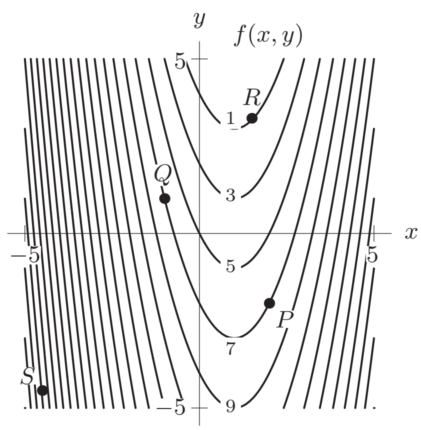


Figure 9.3

1. At the point Q, which of the following is true?

   (a) $f_x > 0, f_y > 0$ (b) $f_x > 0, f_y < 0$
   (c) $f_x < 0, f_y > 0$ (d) $f_x < 0, f_y < 0$

   ANSWER:

   (d), since both are negative.

2. List the points P, Q, R in order of increasing f_x.

   (a) $P > Q > R$ (b) $P > R > Q$ (c) $R > P > Q$
   (d) $R > Q > P$ (e) $Q > R > P$

   ANSWER:

   (b), since $f_x(P) > 0$, $f_x(R) > 0$, and $f_x(P) > f_x(R)$. Also $f_x(Q) < 0$.

3. Use the level curves of $f(x, y)$ in Figure 9.4 to estimate

(a) $f_x(2, 1)$ (b) $f_y(1, 2)$

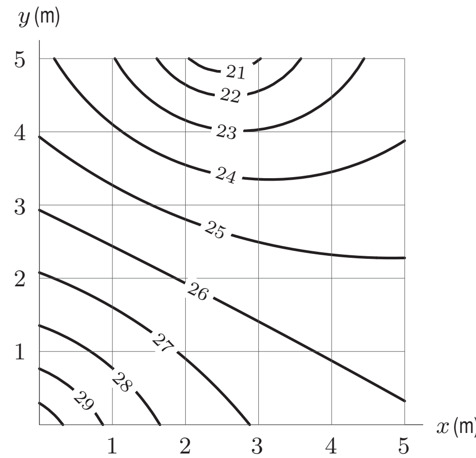


Figure 9.4

ANSWER:

(a) We have

$$f_x(2, 1) \approx \frac{26 - 27}{3.7 - 2} = -\frac{1}{1.7} = -0.588.$$

(b) We have

$$f_y(1, 2) \approx \frac{26 - 26.5}{2.5 - 2} = -1.$$

4. Figure 9.5 shows level curves of $f(x, y)$. At which of the following points is one or both of the partial derivatives, f_x, f_y, approximately zero? Which one(s)?

(a) $(1, -0.5)$
(b) $(-0.4, 1.5)$
(c) $(1.5, -0.4)$
(d) $(-0.5, 1)$

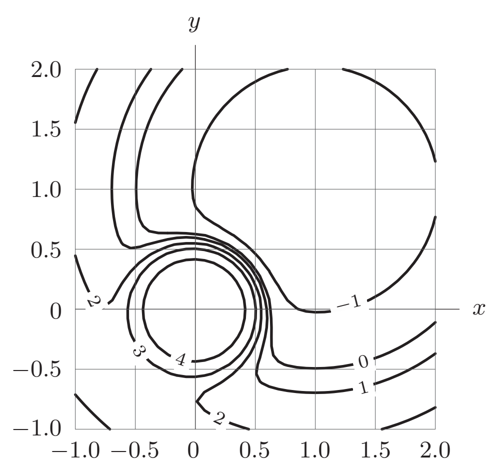


Figure 9.5

ANSWER:

(a) At $(1, -0.5)$, the level curve is approximately horizontal, so $f_x \approx 0$, but $f_y \not\approx 0$.
(b) At $(-0.4, 1.5)$, neither are approximately zero.
(c) At $(1.5, -0.4)$, neither are approximately zero.
(d) At $(-0.5, 1)$, the level curve is approximately vertical, so $f_y \approx 0$, but $f_x \not\approx 0$.

COMMENT:
Ask students to estimate the nonzero partial derivative at one or more of the points.
Ask students to speculate about the partial derivatives at $(0, 0)$.

5. Figure 9.6 is a contour diagram for $f(x, y)$ with the x and y axes in the usual directions. At the point P, is $f_x(P)$ positive or negative? Increasing or decreasing as we move in the the direction of increasing x?

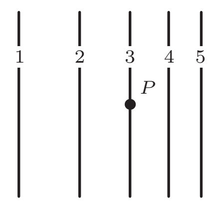


Figure 9.6

ANSWER:
At P, the values of f are increasing at an increasing rate as we move in the positive x-direction, so $f_x(P) > 0$, and $f_x(P)$ is increasing.
COMMENT:
Ask about how f_x changes in the y-direction and about f_y.

6. Figure 9.7 is a contour diagram for $f(x, y)$ with the x and y axes in the usual directions. At the point P, if x increases, what is true of $f_x(P)$? If y increases, what is true of $f_y(P)$?

(a) Have the same sign and both increase.
(b) Have the same sign and both decrease.
(c) Have opposite signs and both increase.
(d) Have opposite signs and both decrease.
(e) None of the above.

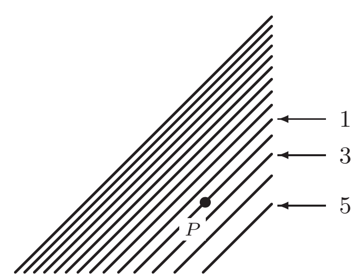


Figure 9.7

ANSWER:
We have $f_x(P) > 0$ and decreasing, and $f_y(P) < 0$ and decreasing, so the answer is (d).

ConcepTests for Section 9.4

1. Let $f(x,y) = x^2\ln(x^2y)$. Which of the following are the two partial derivatives of f? Which is which?

   (a) $2x\ln(x^2y) + 2x, \dfrac{2x}{y}$
   (b) $\dfrac{1}{y}, 2x\ln(x^2y) + \dfrac{2x}{y}$
   (c) $\dfrac{x^2}{y}, 2x + 2x\ln(x^2y)$
   (d) $\dfrac{2x^3}{y} + 2x\ln(x^2y), \dfrac{x^2}{y}$
   (e) $\dfrac{2x}{y} + 2x\ln(x^2y), \dfrac{1}{y}$
   (f) $\dfrac{2x}{y}, 2x\ln(x^2y) + \dfrac{2x^3}{y}$

   ANSWER:
   The partial derivatives are

$$f_x(x,y) = 2x\ln(x^2y) + x^2 \cdot \frac{1}{x^2y} \cdot 2xy = 2x\ln(x^2y) + 2x$$

$$f_y(x,y) = x^2 \cdot \frac{1}{x^2y} \cdot x^2 = \frac{x^2}{y}.$$

   Thus, the answer is (c), with f_y first, f_x second.

2. Which of the following functions satisfy the following equation (called Euler's Equation):

$$xf_x + yf_y = f?$$

   (a) x^2y^3 (b) $x + y + 1$ (c) $x^2 + y^2$ (d) $x^{0.4}y^{0.6}$

   ANSWER:
   Calculate the partial derivatives and check. The answer is (d), and

$$x\frac{\partial}{\partial x}(x^{0.4}y^{0.6}) = x0.4x^{-0.6}y^{0.6} = 0.4x^{0.4}y^{0.6}$$

$$y\frac{\partial}{\partial y}(x^{0.4}y^{0.6}) = yx^{0.4}0.6y^{-0.4} = 0.6x^{0.4}y^{0.6}.$$

   Thus

$$xf_x + yf_y = 0.4x^{0.4}y^{0.6} + 0.6x^{0.4}y^{0.6} = x^{0.4}y^{0.6} = f.$$

3. Figure 9.8 shows level curves of $f(x,y)$. What are the signs of $f_{xx}(P)$ and $f_{yy}(P)$?

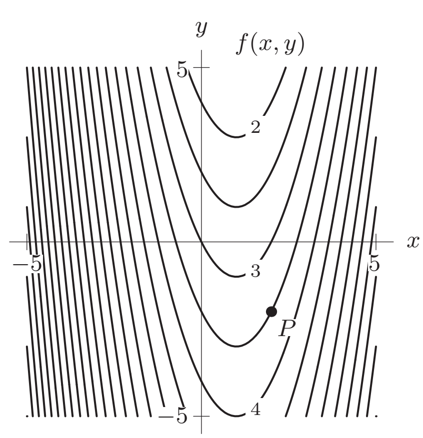


Figure 9.8

ANSWER:

Since f_x is positive and increasing at P, we have $f_{xx}(P) > 0$. Since f_y is negative, but getting less negative at P, we have $f_{yy}(P) > 0$. However, f_y is changing very slowly, so it would also be reasonable to say $f_{yy}(P) \approx 0$.

COMMENT:

As an extension, ask students to find the sign of $f_{xy}(P)$.

4. Figure 9.9 shows level curves of $f(x, y)$. What are the signs of $f_{xx}(Q)$ and $f_{yy}(Q)$?

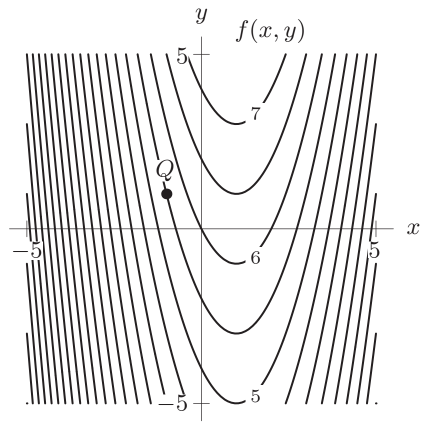


Figure 9.9

ANSWER:

Since f_x is positive and decreasing at Q, we have $f_{xx}(Q) < 0$. Since f_y is positive and decreasing at Q, we have $f_{yy}(Q) < 0$. However, f_y is changing very slowly, so it would also be reasonable to say $f_{yy}(Q) \approx 0$.

COMMENT:

As an extension, ask students to find the sign of $f_{xy}(Q)$.

5. Figure 9.10 shows the temperature T°C as a function of distance x in meters along a wall and time t in minutes. Choose the correct statement and explain your choice without computing these partial derivatives.

(a) $\dfrac{\partial T}{\partial t}(t, 10) < 0$ and $\dfrac{\partial^2 T}{\partial t^2}(t, 10) < 0$.
(b) $\dfrac{\partial T}{\partial t}(t, 10) > 0$ and $\dfrac{\partial^2 T}{\partial t^2}(t, 10) > 0$.
(c) $\dfrac{\partial T}{\partial t}(t, 10) > 0$ and $\dfrac{\partial^2 T}{\partial t^2}(t, 10) < 0$.
(d) $\dfrac{\partial T}{\partial t}(t, 10) < 0$ and $\dfrac{\partial^2 T}{\partial t^2}(t, 10) > 0$.

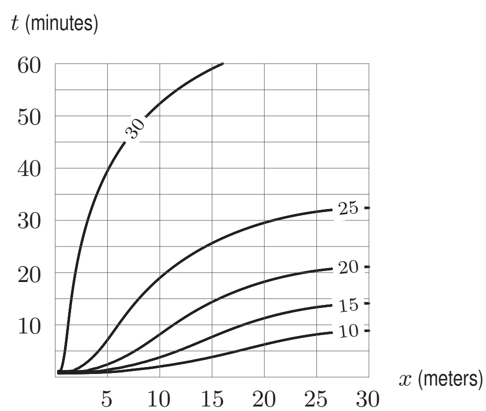


Figure 9.10

ANSWER:

(c). From the graph of the contour lines, we can see that along the line $x = 10$, the temperature T is increasing, thus $\dfrac{\partial T}{\partial t}(t, 10) > 0$. Since the distance between the contour lines is increasing as time increases along the line $x = 10$, the rate of change of $\dfrac{\partial T}{\partial t}(t, 10)$ is decreasing, so $\dfrac{\partial^2 T}{\partial t^2}(t, 10) < 0$.

ConcepTests for Section 9.5

1. Estimate the global maximum and minimum of the functions whose level curves are in Figure 9.11. How many times does each occur?
   (a) Max $= 6$, occurring once; min $= -6$, occurring once
   (b) Max $= 6$, occurring once; min $= -6$, occurring twice
   (c) Max $= 6$, occurring twice; min $= -6$, occurring twice
   (d) Max $= 6$, occurring three times; min $= -6$, occurring three times
   (e) None of the above

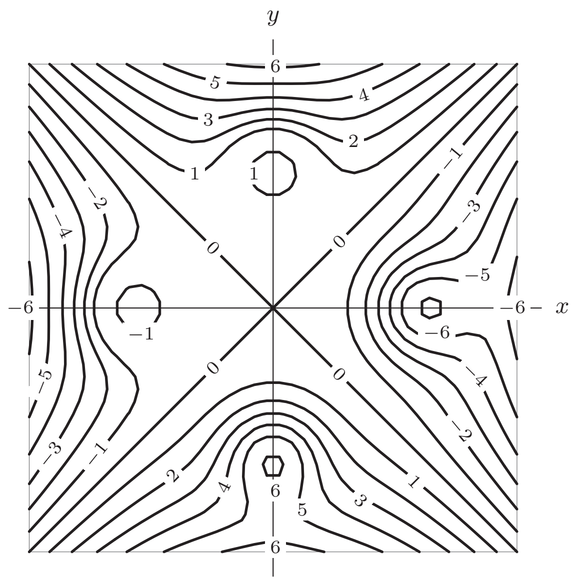


Figure 9.11

ANSWER:
The global max is about 6, or slightly higher. It occurs three times, twice on the negative y-axis and once on the positive y-axis. The global min is -6, or slightly lower. It occurs three times, twice on the positive x-axis and once on the negative x-axis. The answer is (d).

Problems 2–4 refer to the functions $f(x, y)$:

(a) $x^2 + 2y^3$ (b) $x^2y + 4xy + 4y$
(c) $x^2y^3 - x^4 + 2y$ (d) $x \cos y$

2. Which of these functions has a critical point at the origin?
   ANSWER:
   (a). Taking partial derivatives gives $f_x = 2x$ and $f_y = 6y^2$, so $x = 0$ and $y = 0$ give a critical point. The other functions do not have a critical point at the origin.
3. True or False? Function (b) has a local maximum at the origin.
   ANSWER:
   False. Since $f_y(0, 0) = 4$, the origin is not a critical point of f, so it cannot be a local maximum.
4. Which of the functions does not have a critical point?
   ANSWER:
   (c). Taking partial derivatives gives $f_x = 2xy^3 - 4x^3$ and $f_y = 3x^2y^2 + 2$. Because f_y is always greater than or equal to 2, there are no critical points.

Problems 5–8 refer to the functions $f(x, y)$:

(a) $x^2 + 2x + 2y^3 - y^2$ (b) $x^2y + xy$
(c) $x^2y^2 - (1/2)x^4 + 2y$ (d) $x^4y - 7y$

5. Which of the functions has a critical point at the origin?
ANSWER:
(b). Taking partial derivatives gives $f_x = 2xy + y = (2x+1)y$ and $f_y = x^2 + x = (x+1)x$, so $x = 0$ and $y = 0$ give a critical point. The others do not have a critical point at the origin.

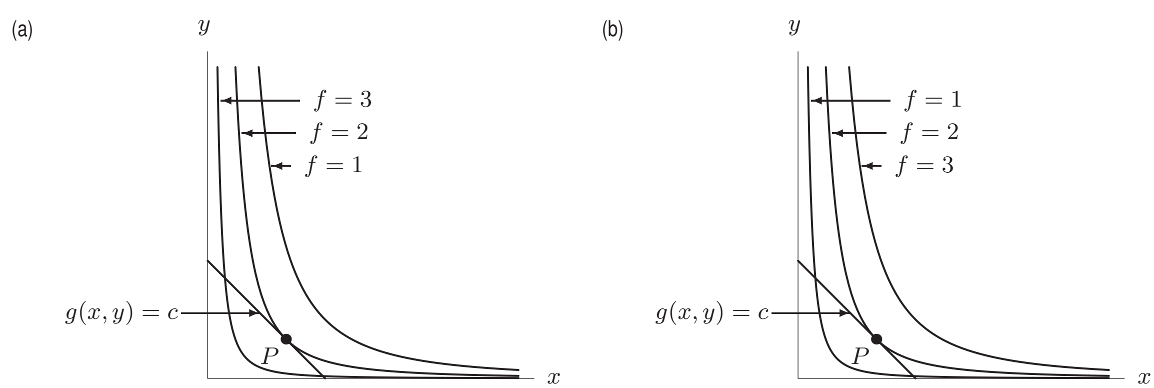
6. True or false? For the function (b), the second derivative test shows that critical point $(0, 0)$ is a local maximum.
ANSWER:
False. Taking partial derivatives gives $f_{xx} = 2y$, $f_{yy} = 0$, and $f_{xy} = 2x + 1$, so
$$D = (2 \cdot 0) \cdot 0 - (2 \cdot 0 + 1)^2 = -1,$$
so the second derivative test tells us $(0, 0)$ is a saddle point.

7. Which of these functions does not have a critical point with $y = 0$?
ANSWER:
(c). Taking partial derivatives gives $f_x = 2xy^2 - 2x^3 = 2x(y^2 - x^2)$ and $f_y = 2x^2y + 2 = 2(x^2y + 1)$. At critical points, we need $y^2 = x^2$ and $x^2y = -1$, so $y^3 = -1$, or $y = -1$. Thus, x may equal either 1 or -1, so the only two critical points are $(1, -1)$ and $(-1, -1)$.

8. Which of these functions does not have a critical point with $x = -1$?
ANSWER:
(d). Taking partial derivatives gives $f_x = 4x^3y$ and $f_y = x^4 - 7$, so the two critical points are $x = \pm 7^{1/4}$, $y = 0$.

ConcepTests for Section 9.6

1. The point P is a maximum or minimum of the function f subject to the constraint $g(x, y) = x + y = c$, with $x, y \geq 0$. For the graphs (a) and (b), does P give a maximum or a minimum of f? What is the sign of λ? If P gives a maximum, where does the minimum of f occur? If P gives a minimum, where does the maximum of f occur?



ANSWER:
(a) The point P gives a minimum; the maximum is at one of the end points of the line segment (either the x- or the y-intercept). The value of λ is negative, since f decreases in the direction in which g increases.
(b) The point P gives a maximum; the minimum is at the x- or y-intercept. The value of λ is positive, since f and g increase in the same direction.

2. Figure 9.12 shows the optimal point (marked with a dot) in three optimization problems with the same constraint. Arrange the corresponding values of λ in increasing order. (Assume λ is positive.)

   (a) I<II<III (b) II<III<I (c) III<II<I (d) I<III<II

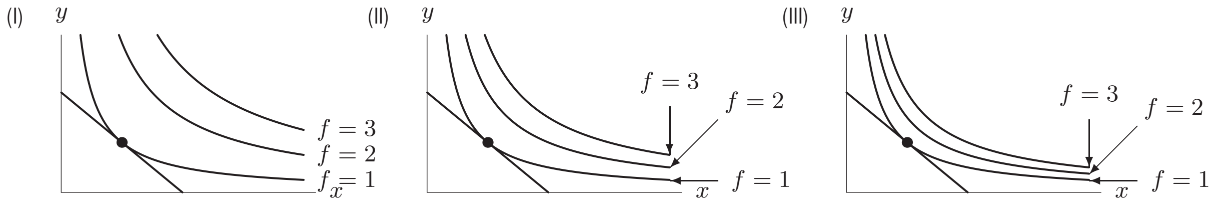


Figure 9.12

ANSWER:
Since λ is the additional f that is obtained by relaxing the constraint by 1 unit, λ is larger if the level curves of f are close together near the optimal point. The answer is I<II<III, so (a).

For Problems 3–4, use Figure 9.13. The grid lines are one unit apart.

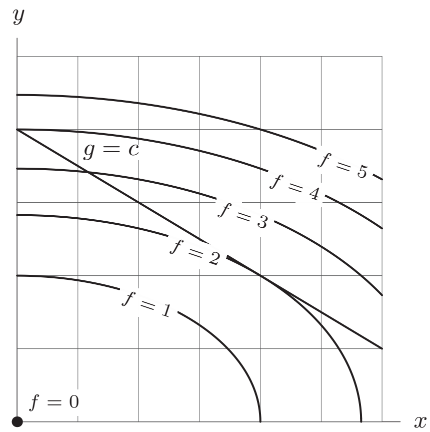


Figure 9.13

3. Find the maximum and minimum values of f on $g = c$. At which points do they occur?
   ANSWER:
   The maximum is $f = 4$ and occurs at $(0, 4)$. The minimum is $f = 2$ and occurs at about $(4, 2)$.
4. Find the maximum and minimum values of f on the triangular region below $g = c$ in the first quadrant.
   ANSWER:
   The maximum is $f = 4$ and occurs at $(0, 4)$. The minimum is $f = 0$ and occurs at the origin.

Problems 5–6 concern the following:

(a) Maximize x^2y^2 subject to $x^2 + y^2 = 4$.
(b) Maximize $x + y$ subject to $x^2 + y^2 = 4$.
(c) Minimize $x^2 + y^2$ subject to $x^2y^2 = 4$.
(d) Maximize x^2y^2 subject to $x + y = 4$.

5. Which one of (a)–(d) does not have its solution at the same point in the first quadrant as the others?

ANSWER:

The answer is (d).

(a) We solve

$$\begin{aligned} 2xy^2 &= 2x\lambda \\ 2x^2y &= 2y\lambda \\ x^2 + y^2 &= 4, \end{aligned}$$

giving $x^2 = y^2$, so $2x^2 = 4$, so (since we are working in the first quadrant) $x = y = \sqrt{2}$.

(b) We solve

$$\begin{aligned} 1 &= 2x\lambda \\ 1 &= 2y\lambda \\ x^2 + y^2 &= 4, \end{aligned}$$

giving $x = y$, so $2x^2 = 4$, so (since we are working in the first quadrant) $x = y = \sqrt{2}$.

(c) We solve

$$\begin{aligned} 2x &= 2xy^2\lambda \\ 2y &= 2x^2y\lambda \\ x^2y^2 &= 4, \end{aligned}$$

giving $x^2 = y^2$, so $(x^2)^2 = 4$, so (since we are working in the first quadrant) $x = y = \sqrt{2}$.

(d) We solve

$$\begin{aligned} 2xy^2 &= \lambda \\ 2x^2y &= \lambda \\ x + y &= 4. \end{aligned}$$

Dividing the first two equations gives $x = y$, so (since we are working in the first quadrant) $x = y = 2$.

Thus, (a), (b), and (c) have their solution at $x = y = \sqrt{2}$, and (d) has its solution at a different point, namely $x = y = 2$.

6. Which two of (a)–(d) have the same extreme value of the objective function in the first quadrant?

ANSWER:

(a) We have $f(\sqrt{2}, \sqrt{2}) = (\sqrt{2})^2(\sqrt{2})^2 = 4$.
(b) We have $f(\sqrt{2}, \sqrt{2}) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$.
(c) We have $f(\sqrt{2}, \sqrt{2}) = (\sqrt{2})^2 + (\sqrt{2})^2 = 4$.
(d) We have $f(2, 2) = 2^2 \cdot 2^2 = 16$.

Thus, (a) and (c) have the same extreme value.

7. Find the maximum of the production function $f(x, y) = xy$ in the first quadrant subject to each of the three budget constraints.

I. $x + y = 12$ II. $2x + 5y = 12$ III. $3x + y/2 = 12$

Arrange the x- and y-coordinates of the optimal point in increasing order. Pick one of (a)–(e) and one of (f)–(j).

(a) x: I < II < III
(b) x: III < II < I
(c) x: II < III < I
(d) x: II < I < III
(e) x: III < I < II

(f) y: I < II < III
(g) y: III < II < I
(h) y: II < III < I
(i) y: II < I < III
(j) y: III < I < II

ANSWER:
For (I), we have

$$\begin{aligned} y &= \lambda \\ x &= \lambda \\ x + y &= 12, \end{aligned}$$

so $x = y$ and the solution is $x = 6$, $y = 6$.
For (II), we have

$$\begin{aligned} y &= 2\lambda \\ x &= 5\lambda \\ 2x + 5y &= 12, \end{aligned}$$

so $2x = 5y$, and the solution is $x = 3, y = 1.2$.
For (III), we have

$$\begin{aligned} y &= 3\lambda \\ x &= \lambda/2 \\ 3x + y/2 &= 12, \end{aligned}$$

so $3x = y/2$, and the solution is $x = 2, y = 12$.
Thus, for x: III<II<I
for y: II<I<III
The answer is (b) and (i).