Chapter One

ConcepTests for Section 1.1

1. As a person hikes down from the top of a mountain, the variable T represents the time, in minutes, since the person left the top of the mountain, and the variable H represents the height, in feet, of the person above the base of the mountain. Table 1.1 gives values at several different times for these variables.

Table 1.1

Time T	20	30	40	50	60
Height H	1000	810	730	810	580

Which of the following statements is true?

(a) T is a function of H
(b) H is a function of T
(c) Both statements are true: T is a function of H and H is a function of T
(d) Neither statement is true: T is not a function of H and H is not a function of T

ANSWER:

(b) We see that H is a function of T since for every value of T, there is a unique value of H. As the person hikes down the mountain, the height at any given moment in time is uniquely determined. We know that T is not a function of H, since we see in Table 1.1 that, for example, the height $H = 810$ corresponds to at least two values of T.

2. As a person hikes down from the top of a mountain, the variable T represents the time, in minutes, since the person left the top of the mountain, and the variable H represents the height, in feet, of the person above the base of the mountain. We have $H = f(T)$. The statement $f(100) = 300$ means:

(a) The mountain rises 300 feet above its base, and it takes 100 minutes to descend from the top of the mountain.
(b) The mountain rises 100 feet from its base and it takes 300 minutes to descend from the top of the mountain.
(c) At a time of 100 minutes after leaving the top of the mountain, the person is 300 feet above the base of the mountain.
(d) At a time of 300 minutes after leaving the top of the mountain, the person is 100 feet above the base of the mountain.

ANSWER:

(c) The statement $f(100) = 300$ tells us that when $T = 100$, we have $H = 300$. Therefore, (c) is the correct answer.

3. As a person hikes down from the top of a mountain, the variable T represents the time, in minutes, since the person left the top of the mountain, and the variable H represents the height, in feet, of the person above the base of the mountain. We have $H = f(T)$. The vertical intercept for the graph of this function represents:

(a) The time it takes the person to descend from the top of the mountain to the base of the mountain.
(b) The height of the person in feet above the base of the mountain when the person is at the top of the mountain.
(c) The height of the person in feet above the base of the mountain, as the person hikes down the mountain.
(d) The time when the person begins to descend down the mountain.

ANSWER:

(b) Since $H = f(T)$, the vertical intercept is the value of H when $T = 0$. Since $T = 0$ means the person is at the top of the mountain, we want the value of H when the person is at the top of the mountain, which is answer (b).

4. As a person hikes down from the top of a mountain, the variable T represents the time, in minutes, since the person left the top of the mountain, and the variable H represents the height, in feet, of the person above the base of the mountain. We have $H = f(T)$. The horizontal intercept for the graph of this function represents:

(a) The time it takes the person to descend from the top of the mountain to the base of the mountain.
(b) The height of the person in feet above the base of the mountain when the person is at the top of the mountain.
(c) The height of the person in feet above the base of the mountain, as the person hikes down the mountain.
(d) The time when the person begins to descend down the mountain.

ANSWER:

(a) Since $H = f(T)$, the horizontal intercept is the value of T when $H = 0$. Since $H = 0$ means the person is at the base of the mountain, we want the value of T when the person reaches the base of the mountain, which is answer (a).

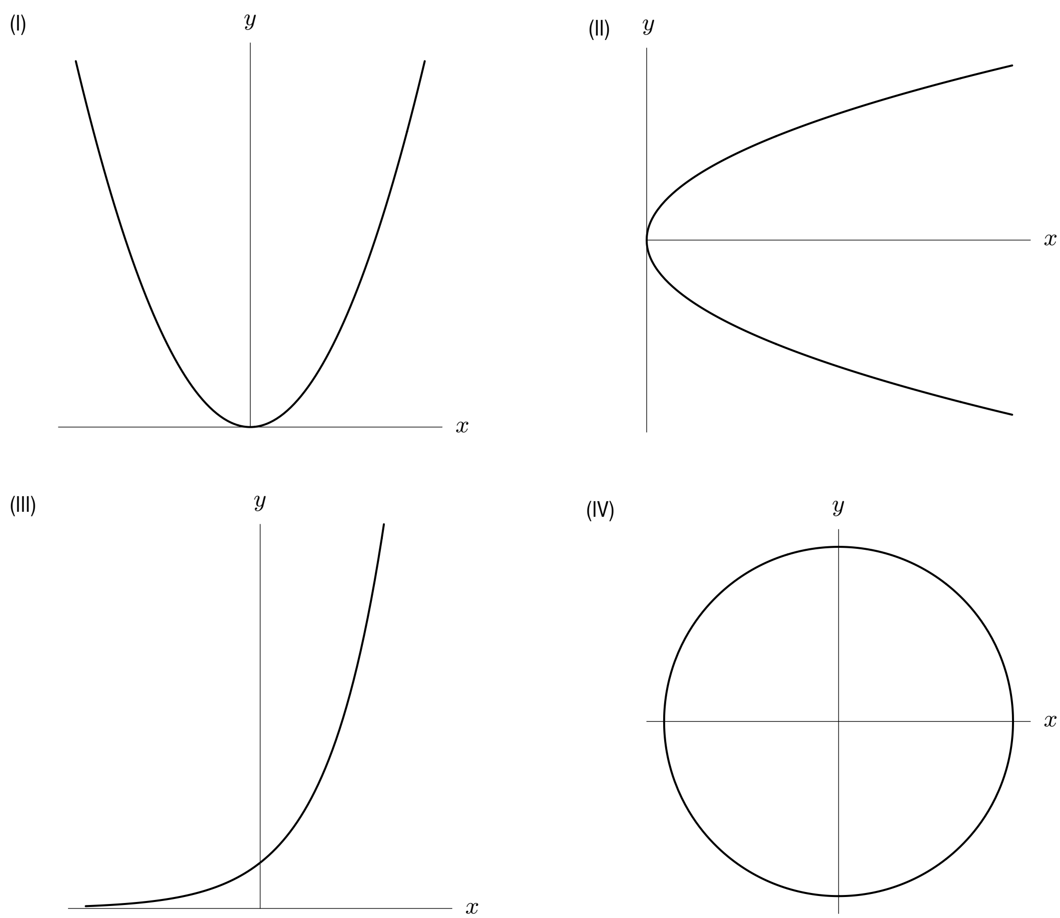
5. A patient's heart rate, R, in beats per minute, is a function of the dose, D of a drug, in mg. We have $R = f(D)$. The statement $f(50) = 70$ means:
   (a) The patient's heart rate goes from 70 beats per minute to 50 beats per minute when a dose is given.
   (b) When a dose of 50 mg is given, the patient's heart rate is 70 beats per minute.
   (c) The dose ranges from 50 mg to 70 mg for this patient.
   (d) When a dose of 70 mg is given, the patient's heart rate is 50 beats per minute.

   ANSWER:
   (b) Since $R = f(D)$, the statement $f(50) = 70$ means that when $D = 50$, we have $R = 70$, so the answer is (b).

6. A patient's heart rate, R, in beats per minute, is a function of the dose, D of a drug, in mg. We have $R = f(D)$. The vertical intercept for the graph of this function represents:
   (a) The maximum dose of the drug.
   (b) The maximum heart rate.
   (c) The dose of the drug at which the patient's heart stops beating.
   (d) The patient's heart rate if none of the drug is administered.

   ANSWER:
   (d) Since $R = f(D)$, the vertical intercept is the value of R when $D = 0$. This is the heart rate when the dose of the drug is 0, so the answer is (d).

7. In which of graphs (I)–(IV) could y be a function of x?
   (a) I only
   (b) II only
   (c) III only
   (d) IV only
   (e) I and II only
   (f) I and III only
   (g) II and IV only
   (h) All of them



ANSWER:

(f) We use the vertical line test to determine whether a graph represents a function. In graphs I and III, all vertical lines cross the curve at most once, while in II and IV, some vertical lines cross the curve more than once. Graphs I and III may be graphs of functions, while graphs II and IV definitely are not.

8. There are three poles spaced 10 meters apart as shown in Figure 1.1. Joey walks from pole C to pole B, stands there for a short time, then runs to pole C, stands there for a short time and then jogs to pole A. Which of the following graphs describes Joey's distance from pole A?

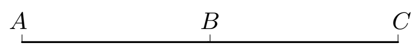
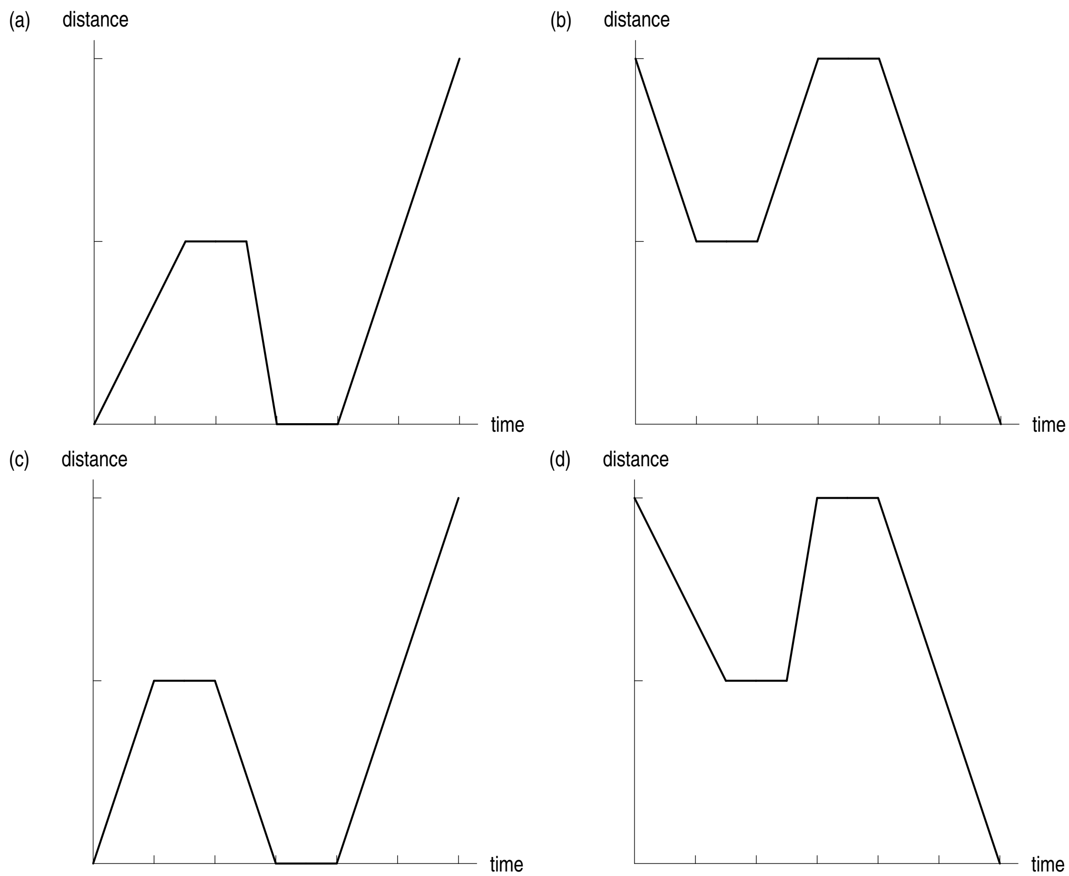


Figure 1.1



ANSWER:

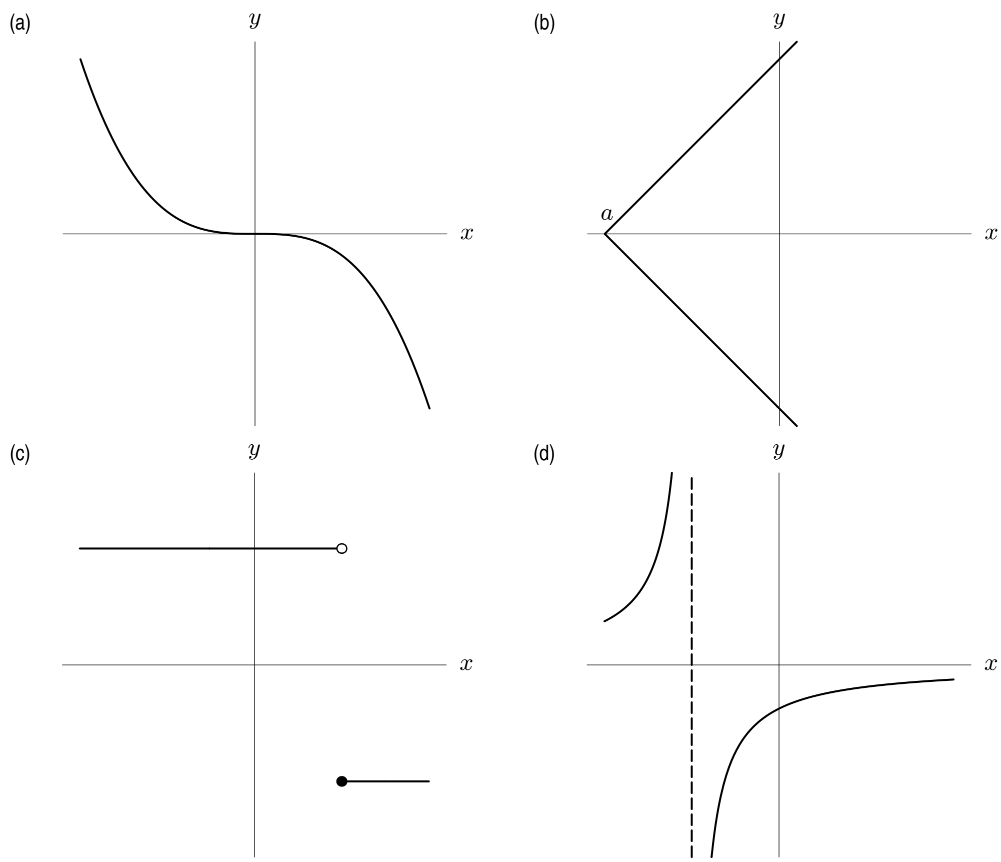
(d). The graph describes the distance from pole A, so it cannot start at the origin. This eliminates choices (a) and (c). Joey will spend more time walking between poles than running. This eliminates (b).

COMMENT:

You can also reason using slopes ($|\text{slope}| = $ speed) as follows: In the first interval he is walking, then he is going three times as fast (running), and finally he is jogging (half the speed of running).

Follow-up Question. Give a scenario that describes the remaining graphs.

9. Which of the graphs does not represent y as a function of x?



ANSWER:

(b). For $x > a$ there are two function values corresponding to the same value of x.

COMMENT:

It may be worth noting that we are assuming that the entire graph is shown—that the function does not behave differently outside the given window.

10. Which of the following functions has its domain identical with its range?

(a) $f(x) = x^2$
(b) $g(x) = \sqrt{x}$
(c) $h(x) = x^3$
(d) $i(x) = |x|$

ANSWER:

(b) and (c). For $g(x) = \sqrt{x}$, the domain and range are all nonnegative numbers, and for $h(x) = x^3$, the domain and range are all real numbers.

COMMENT:

It is worth considering the domain and range for all choices.

ConcepTests for Section 1.2

1. It costs a total of C dollars to extract T tons of ore from a copper mine. If C is a linear function of T, the units of the slope of the line are:

   (a) Tons
   (b) Dollars
   (c) Tons/dollar
   (d) Dollars/ton

   ANSWER:
   (d) Since $C = f(T)$, the slope is $\Delta C/\Delta T$, so the units are C-units over T-units, or dollars/ton.

2. Assume $y = 100 - 2x$. If x goes up by 3, the corresponding y-value changes by:

   (a) 300
   (b) -300
   (c) 6
   (d) -6
   (e) 94
   (f) -94

   ANSWER:
   (d) The slope, -2, tells us the change in y for every unit increase in x, so if x goes up by 3, we see that y changes by $-2 \cdot 3 = -6$.

3. When a person goes into shock, the cardiac output, in liters of blood per minute, decreases. One person's cardiac output is 12 liters per minute when the person first goes into shock, and decreases by 2 liters per minute every hour that the person is in shock. Write a formula for cardiac output C as a function of t, the time in hours since a person first went into shock.

   (a) $C = 12 - 2t$
   (b) $t = 12 - 2C$
   (c) $C = -2 + 12t$
   (d) $t = -2 + 12t$
   (e) $C = 12 + 2t$
   (f) $t = 12 + 2C$

   ANSWER:
   (a) This is a linear function since the change is constant over every equal time interval. The cardiac output starts at 12 when $t = 0$, so the vertical intercept is 12. The cardiac output changes by -2 for every additional hour, so the slope is -2.

4. The value of a refrigerator, V (in dollars), is a function of the age of the refrigerator, a in years. We have $V = 1200 - 200a$. Give the units and meaning of the vertical intercept.

   (a) 1200 dollars/year, represents the change in the value as the refrigerator ages one year.
   (b) -200 dollars/year, represents the change in the value as the refrigerator ages one year.
   (c) 1200 dollars, represents the value of the refrigerator when it is new.
   (d) -200 dollars, represents the value of the refrigerator when it is new.
   (e) 6 years, represents the age of the refrigerator when it has lost all of its value.

   ANSWER:
   (c) The vertical intercept is the constant term 1200. It is the value of V when $a = 0$, or the value of the refrigerator when it is 0 years old.
   COMMENT:
   You might point out that answer (e) is the horizontal intercept.

5. The value of a refrigerator, V (in dollars), is a function of the age of the refrigerator, a in years. We have $V = 1200 - 200a$. Give the units and meaning of the slope.

(a) 1200 dollars/year, represents the change in the value as the refrigerator ages one year.
(b) -200 dollars/year, represents the change in the value as the refrigerator ages one year.
(c) 1200 dollars, represents the value of the refrigerator when it is new.
(d) -200 dollars, represents the value of the refrigerator when it is new.
(e) 6 years, represents the age of the refrigerator when it has lost all of its value.
(f) 1200 years/dollar, represents the change in age as the refrigerator loses value.
(g) -200 years/dollar, represents the change in age as the refrigerator loses value.

ANSWER:

(b) The slope is the coefficient of a, so is -200. It is $\Delta V/\Delta a$, so the units are dollars/year. It represents the change in V when a increases one year.

6. The graph in Figure 1.2 is a representation of which of the following functions?

(a) $y = 6x + 6$
(b) $y = -3x + 6$
(c) $y = -3x + 2$
(d) $y = 6x - 2$

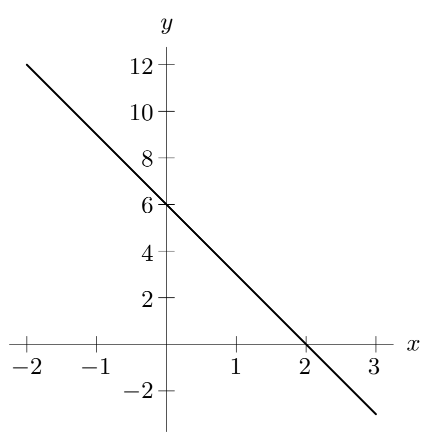


Figure 1.2

ANSWER:

(b). The line has slope -3 and y-intercept 6.

COMMENT:

Other methods of reasoning could be used. For example, the line shown has a negative slope, which eliminates choices (a) and (d). The y-intercepts for choices (b) and (c) are 6 and 2, respectively. From the graph the y-intercept is 6.

7. The graph in Figure 1.3 is a representation of which of the following functions?

   (a) $y = 3x - 2$
   (b) $y = 2x + 1.5$
   (c) $y = 2x + 3$
   (d) $2y = 6x - 3$

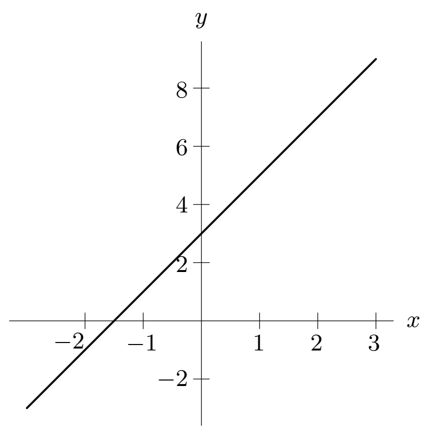


Figure 1.3

ANSWER:
(c). The line has slope 2 and y-intercept 3.
COMMENT:
Note that the y-intercept on the graph is positive, which eliminates choices (a) and (d). The lines given in choices (b) and (c) have the same slope, so the choice will depend on the y-intercept, which appears to be 3, not 1.5.

8. The slope of the line connecting the points $(1, 4)$ and $(3, 8)$ is

   (a) $-\frac{1}{2}$
   (b) -2
   (c) $\frac{1}{2}$
   (d) 2

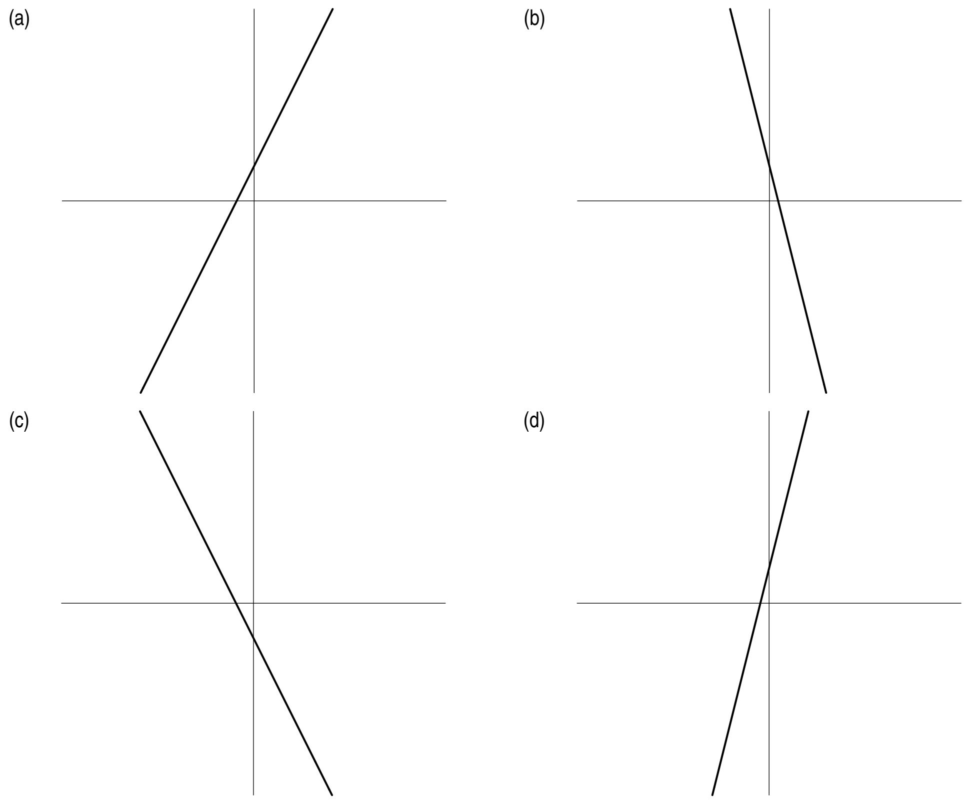
ANSWER:
(d). $\frac{\text{rise}}{\text{run}} = \frac{8-4}{3-1} = \frac{4}{2} = 2.$
COMMENT:
You might point out in finding slopes, the order of the points in the ratio $(y_2 - y_1)/(x_2 - x_1)$ is immaterial.

9. Put the following linear functions in order of increasing slope.

   (a) $y = \pi x + 9$
   (b) $y = 3x + 1$
   (c) $y = -10x$
   (d) $y = x$
   (e) $y = \frac{x}{10} + 7$
   (f) $y = -100$

ANSWER:
(c), (f), (e), (d), (b), (a). In order to put the lines in the correct order, consider the slope of each function.
COMMENT:
This question was used as an elimination question in a classroom session modeled after “Who Wants to be a Millionaire?”, replacing “Millionaire” by “Mathematician”.

10. List the lines in the figure below in the order of increasing slope. (The graphs are shown in identical windows.)



ANSWER:

(b), (c), (a), (d).

COMMENT:

You may want to point out the difference between slope (numerical values) and steepness (absolute value of the slope).

11. Which of the following lines have the same slope?

(a) $y = 3x + 2$
(b) $3y = 9x + 4$
(c) $3y = 2x + 6$
(d) $2y = 6x + 4$

ANSWER:

(a), (b), and (d). Solving for y in the last three choices gives $y = 3x + \frac{4}{3}$, $y = \frac{2}{3}x + 2$, and $y = 3x + 2$. Thus (a), (b), and (d) have the same slope, 3.

COMMENT:

You could point out that (a) and (d) are the same line.

12. Which of the following tables could represent linear functions?

(a)

x	$f(x)$
1	1
2	2
3	4
4	8

(b)

x	$g(x)$
1	-12
2	-9
3	-6
4	-3

(c)

x	$h(x)$
1	10
2	9
4	6
8	3

(d)

x	$j(x)$
1	12
2	14
4	16
8	18

ANSWER:

(b). This is the only table with a constant difference, -3, for the same increase in x. Therefore, (b) is the only one representing a linear function.

COMMENT:

You could point out that (d) fails to be a linear function because the slope between consecutive points is not constant. This happens even though the function values increase by constant amounts because the x values do not increase by constant amounts.

13. Which of the following lines have the same vertical intercept?

(a) $y = 3x + 2$
(b) $3y = 9x + 4$
(c) $3y = 2x + 6$
(d) $2y = 6x + 4$

ANSWER:

(a), (c), and (d). Solving for y in the last three choices gives $y = 3x + \frac{4}{3}$, $y = \frac{2}{3}x + 2$, and $y = 3x + 2$. Thus (a), (c), and (d) have the same vertical intercept, 2.

COMMENT:

You could graph these four equations and then ask about the intercept.

14. Every line has a vertical intercept.

(a) True
(b) False

ANSWER:

(b). The line $x = a$, where $a \neq 0$, does not have a vertical intercept.

COMMENT:

Note that every non-vertical line has a vertical intercept.

15. Every line has a horizontal intercept.

(a) True
(b) False

ANSWER:

(b). The line $y = a$ where $a \neq 0$, does not have a horizontal intercept.

COMMENT:

Note that all lines with a nonzero slope have a horizontal intercept.

16. Every line has both a horizontal intercept and a vertical intercept.

(a) True
(b) False

ANSWER:

(b). Either a horizontal line or a vertical line, excluding the horizontal and vertical axes, provides a counterexample.

COMMENT:

Note that all lines of the form $y = mx + b$, where $m \neq 0$, have both types of intercepts.

17. Every non-horizontal line must have at most one horizontal intercept.

(a) True
(b) False

ANSWER:

(a). Lines of the form $y = mx + b$, where $m \neq 0$, have $-\frac{b}{m}$ as the horizontal intercept, while vertical lines have the form $x = a$, where a is the horizontal intercept.

COMMENT:

Ask the students what this means geometrically.

ConcepTests for Section 1.3

1. Table 1.2 gives sales of the medicinal herb saw palmetto, in millions of dollars, for several different years. [1]

Table 1.2

Year	1997	1998	1999	2000	2001
Sales (million dollars)	85	107	116	122	123

The average rate of change of sales over the period 1997 to 2001 is:

(a) 123 million dollars/year
(b) 123 million dollars
(c) 38 million dollars/year
(d) 38 million dollars
(e) 9.5 million dollars/year
(f) 9.5 million dollars
(g) $4/38$ million dollars/year
(h) $4/38$ million dollars

ANSWER:
(e) The average rate of change of sales per year is change in sales divided by change in year, so we have

$$\text{Rate of change} = \frac{123 - 85}{2001 - 1997} = \frac{38}{4} = 9.5.$$

The units are sales-units over years, or million dollars per year.

2. The rate of change of sales of the medicinal herb saw palmetto in the US during the period 1997 to 2001 is 9.5 million dollars per year. [2] This means that, during the years 1997 to 2001, in the US:

(a) Sales of saw palmetto averaged 9.5 million dollars each year.
(b) Sales of saw palmetto increased by an average of 9.5 million dollars each year.
(c) Sales of saw palmetto were 9.5 million dollars in each of the years.
(d) Sales of saw palmetto went up by 9.5 million dollars in each of the years.

ANSWER:
(b) The rate of change gives the change in sales per year, not the actual sales per year.

3. The gestation time of mammals, in months, is a function of the average adult size, in kilograms, of the mammal. What are the units of average rate of change of gestation time with respect to size?

(a) Months
(b) Kilograms
(c) Kilograms per month
(d) Months per kilogram

ANSWER:
(d) The rate of change of gestation time with respect to size is the change in gestation time divided by the change in size, so the units are months per kilogram.

4. The number of acres in a region cleared for farming follows the formula $A = f(t) = 2t^2$ where t is the number of months since the region started to be farmed and t ranges from $t = 0$ to $t = 10$. Find the average rate of change in the number of acres cleared for farming between $t = 1$ and $t = 4$. Give units with your answer.

(a) 10 acres/year
(b) 30 acres
(c) 10 years/acre
(d) 30 years
(e) 0.10 years/acre
(f) 0.10 acres/year

ANSWER:
(a) We have

$$\text{Rate of change} = \frac{f(4) - f(1)}{4 - 1} = \frac{32 - 2}{3} = 10.$$

The units are A-units over t-units, or acres per year.

[1] The World Almanac and Book of Facts 2003, p. 95.
[2] The World Almanac and Book of Facts 2003, p. 95.

For Problems 5–8, use the graph of $y = f(x)$ in Figure 1.4.

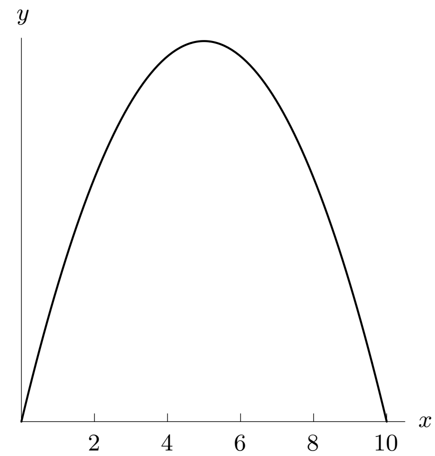


Figure 1.4

5. The rate of change of y between $x = 1$ and $x = 3$ is:
   (a) Positive
   (b) Negative
   (c) Zero

   ANSWER:

   (a) The average rate of change is represented graphically by the slope of a secant line. We see that the secant line between $t = 1$ and $t = 3$ has a positive slope.

6. The rate of change of y between $x = 4$ and $x = 7$ is:
   (a) Positive
   (b) Negative
   (c) Zero

   ANSWER:

   (b) The average rate of change is represented graphically by the slope of a secant line. We see that the secant line between $t = 4$ and $t = 7$ has a negative slope.

7. The rate of change of y between $x = 4$ and $x = 6$ is:
   (a) Positive
   (b) Negative
   (c) Zero

   ANSWER:

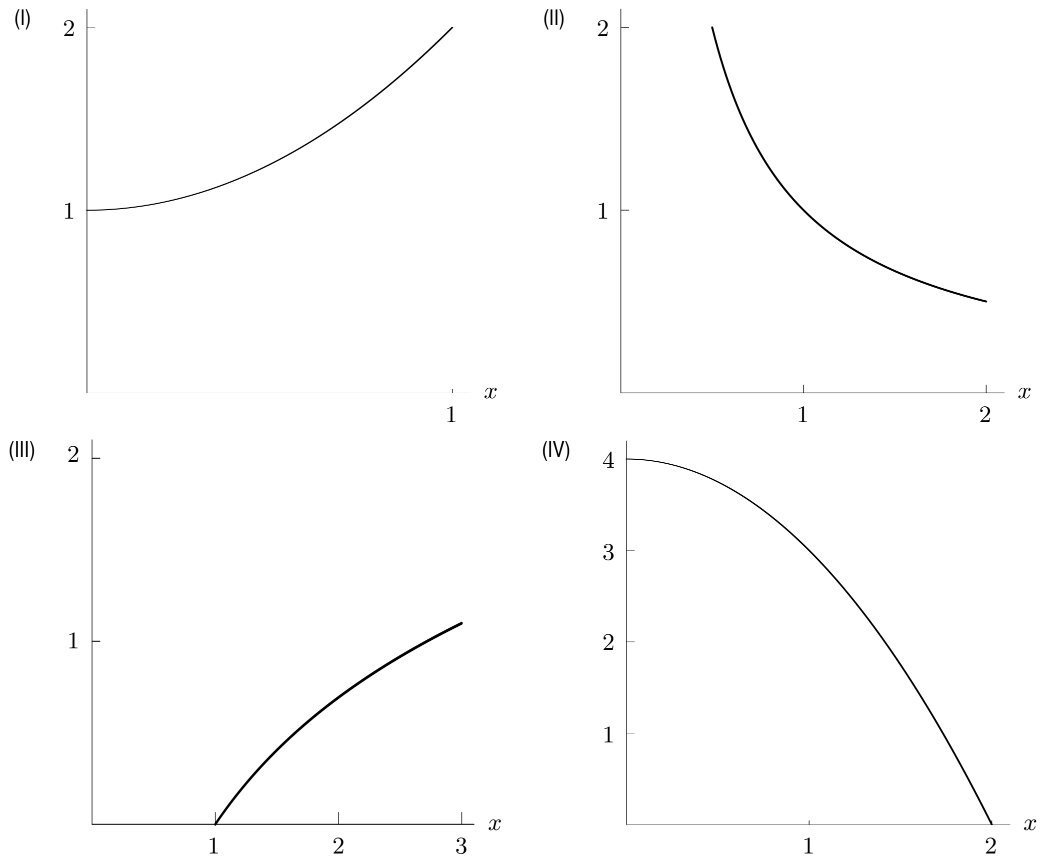
   (c) The average rate of change is represented graphically by the slope of a secant line. We see that the secant line between $t = 4$ and $t = 6$ is horizontal so the slope is zero.

8. Which of the following is the largest of the three quantities?
   (a) The average rate of change of y between $x = 1$ and $x = 2$
   (b) The average rate of change of y between $x = 3$ and $x = 5$
   (c) The average rate of change of y between $x = 4$ and $x = 7$

   ANSWER:

   (a) The average rate of change is represented graphically by the slope of a secant line. We see that the secant line between $t = 1$ and $t = 2$ is steeper than the secant line between $t = 3$ and $t = 5$. Notice also that the slope of secant line between $t = 4$ and $t = 7$ is negative.

For Problems 9–12, use the graphs (I)–(IV).



9. Which graph shows a function that is increasing and concave down?
   ANSWER:
   (III)
   COMMENT:
   Follow-up Question. Is this still true if the graph is shifted up one unit?
10. Which graph shows a function that is increasing and concave up?
   ANSWER:
   (I)
   COMMENT:
   Follow-up Question. Is this still true if the graph is reflected across the y-axis?
11. Which graph shows a function that is decreasing and concave down?
   ANSWER:
   (IV)
   COMMENT:
   Follow-up Question. Is this still true if the graph is reflected across the x-axis?
12. Which graph shows a function that is decreasing and concave up?
   ANSWER:
   (II)
   COMMENT:
   Follow-up Question. Is this still true if the vertical distance from the origin is doubled at every point on the graph?

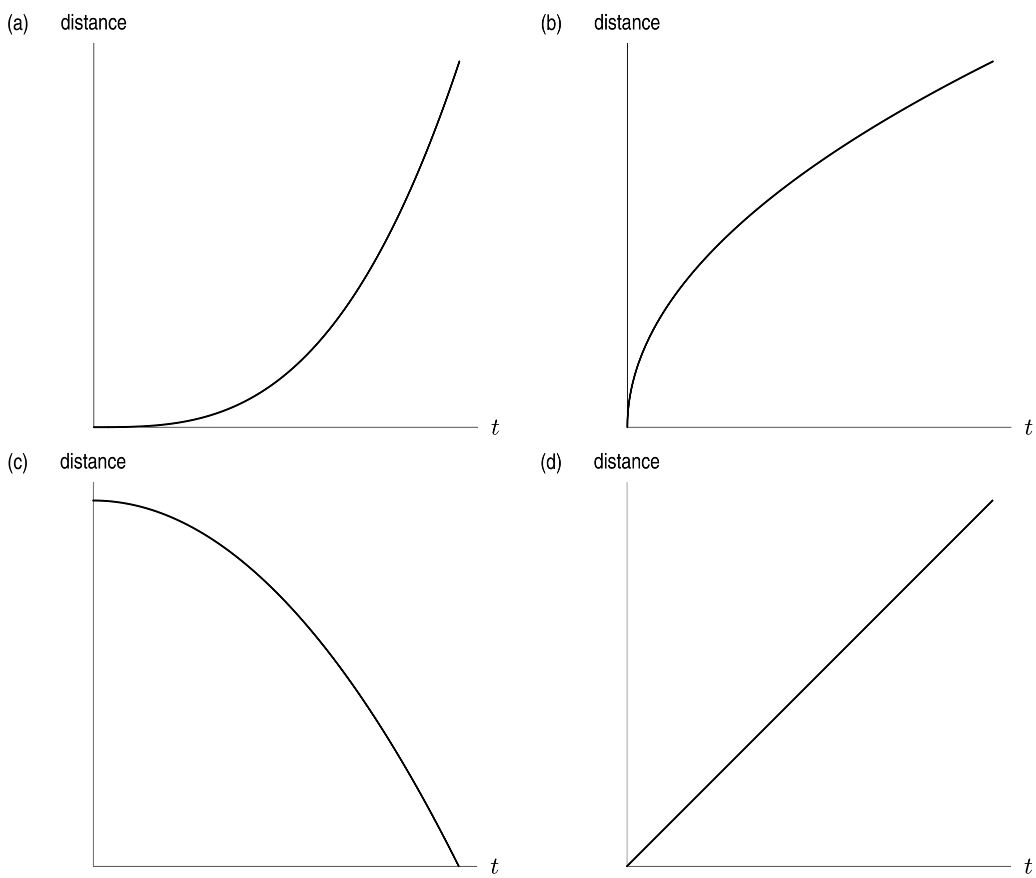
13. Which of the following functions is *not* increasing?
   (a) The elevation of a river as a function of distance from its mouth
   (b) The length of a single strand of hair as a function of time
   (c) The height of a person from age 0 to age 80
   (d) The height of a redwood tree

ANSWER:
(c). In general, people stop growing when they are young adults and, before they are 80, they begin to lose height.
COMMENT:
This question expands a student's idea of a function. You could ask students to supply some more functions that are increasing.

14. Considering y as a function of x, which of the following lines represent decreasing functions?
   (a) $x + y = 2$ (b) $x - y = -2$ (c) $2x - 3y = 6$ (d) $2x + 3y = -6$

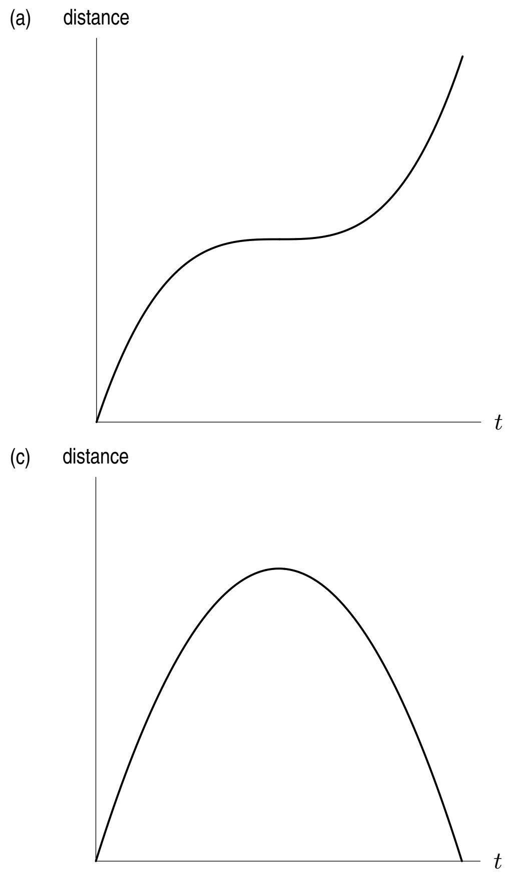
ANSWER:
(a) and (d). Lines with negative slopes are decreasing functions. (a) has a slope of -1 and (d) has a slope of $-\frac{2}{3}$, and thus are decreasing functions.
COMMENT:
These equations could also be graphed to show the slopes geometrically.

15. Which of the graphs represents the position of an object that is slowing down?



ANSWER:
(b). If the object is slowing down, then the changes in distance over various time intervals of the same length will decrease as time increases.
COMMENT:
You could have your students describe how the object is traveling in the other choices.

16. Which of the graphs represents the position of an object that is speeding up and then slowing down?



ANSWER:
(b). The graph has a positive slope everywhere, which is increasing for t near zero and decreasing for larger times.
COMMENT:
You could have students describe how the object is traveling in the other choices.

17. The graph of a function is either concave up or concave down.

(a) True
(b) False

ANSWER:
(b). A function can change concavity or can be a straight line and have no concavity.
COMMENT:
Show the different possibilities.

ConcepTests for Section 1.4

1. One of the graphs in Figure 1.5 is a supply curve and the other is a demand curve. Which is which?
   (a) I is supply and II is demand.
   (b) I is demand and II is supply.

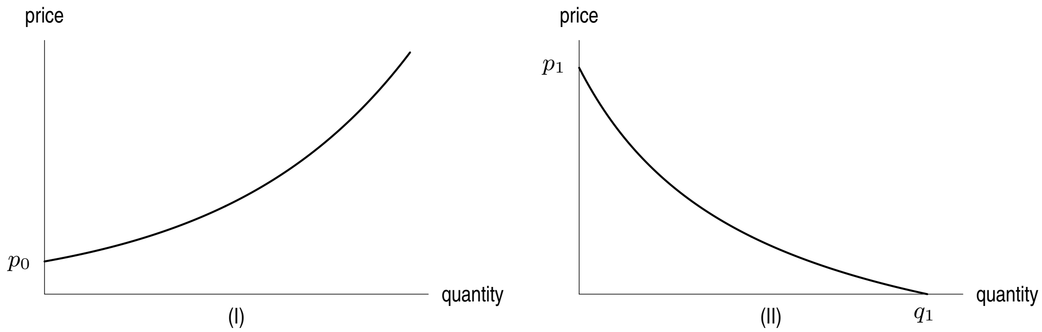


Figure 1.5

ANSWER:
(a) Generally manufacturers produce more when prices are higher. Therefore curve I is the supply curve. Consumers buy less when prices are higher. Therefore curve II is the demand curve.

2. The cost in dollars to produce q tons of an item is given by the cost function

$$C = 100 + 20q.$$

What are the units of the 20?
   (a) Dollars
   (b) Tons
   (c) Dollars/Ton
   (d) Tons/Dollar

ANSWER:
(c) Since the cost C is in dollars, both 100 and $20q$ are in dollars. We have

$$\text{Units of } 20 \cdot \text{Units of } q = \text{dollars}.$$

Since q is in tons,

$$\text{Units of } 20 = \text{dollars/ton}.$$

3. Figure 1.6 shows cost and revenue functions and Figure 1.7 shows supply and demand curves. Which of the points A or B is a break-even point, and which is an equilibrium point?
   (a) A is break-even, and B is equilibrium.
   (b) B is break-even, and A is equilibrium.

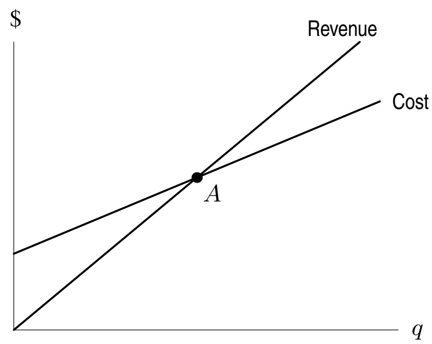


Figure 1.6

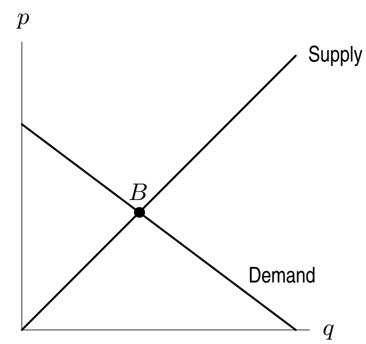


Figure 1.7

ANSWER:

(a). The revenue and cost curves are about a company producing a product. At a lower production level than that given by A, the cost curve is above the revenue curve, so the company loses money. At point A, the company's revenue and cost are equal, so its profit $\pi = R - C$ is zero and the company just breaks even.

4. Supply and demand curves are shown in Figure 1.8. If a tax of \$5 per item sold is imposed on the supplier, which of the points A, B, C, or D could be the new equilibrium?
   (a) A
   (b) B
   (c) C
   (d) D

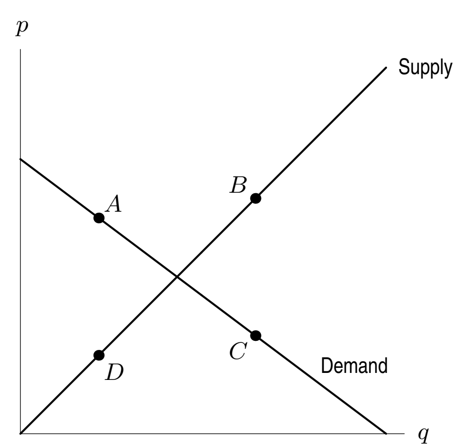


Figure 1.8

ANSWER:

(a) The equilibrium before the tax is imposed is at the intersection of the original supply and demand curves. In the presence of taxes, the new equilibrium is at the intersection of new supply and demand curves. Since the tax is levied on the suppliers, they produce less at each price. Thus the supply curve moves toward the p-axis. The consumers' demand curve is unchanged, so the new equilibrium is at point A on the original demand curve.

Alternatively, notice that the tax raises the supply curve up by \$5 because suppliers have to be paid \$5 more to produce the same quantity. This shifts the equilibrium point up as well, to the point A.

COMMENT:

Notice that the equilibrium price rises with the imposition of the tax, but does not rise by the full \$5 of the tax. Thus the tax burden is shared by the consumer who pays more than before the tax and the supplier who keeps less than before the tax.

ConcepTests for Section 1.5

1. The concentration of a pollutant in a lake is 85 parts per million (ppm) and is increasing at a rate of 4.6% each year. A possible formula for the concentration C as a function of year t is:

(a) $C = 85 + 4.6t$
(b) $C = 85 - 4.6t$
(c) $C = 85 + 0.046t$
(d) $C = 85 - 0.046t$
(e) $C = 85(0.046)^t$
(f) $C = 85(0.954)^t$
(g) $C = 85(1.046)^t$
(h) $C = 85(1.46)^t$
(i) $C = 85(0.46)^t$
(j) $C = 4.6(0.85)^t$

ANSWER:

(g) Since the concentration is increasing by a constant percent each time unit, the function is exponential. The base is $1 + r$ where r is the percent growth rate (written as a decimal). In this case, the base is $1 + 0.046 = 1.046$. The correct answer is $C = 85(1.046)^t$.

2. The amount, A (in mg), of a drug in the body is 25 when it first enters the system is decreases by 12% each hour. A possible formula for A as a function of t, in hours after the drug enters the system, is:

(a) $A = 25 + 12t$
(b) $A = 25 - 12t$
(c) $A = 25 + 0.12t$
(d) $A = 25 - 0.12t$
(e) $A = 25(0.12)^t$
(f) $A = 25(0.88)^t$
(g) $A = 25(1.12)^t$
(h) $A = 25(1.88)^t$
(i) $A = 25(-0.12)^t$
(j) $A = 12(0.25)^t$

ANSWER:

(f) Since the amount is decreasing by a constant percent each time unit, the function is exponential. The base is $1 + r$ where r is the percent growth or decay rate (written as a decimal). In this case, since the amount is decreasing, we know that r is negative, so $r = -0.12$. The base is $1 - 0.12 = 0.88$. The correct answer is $A = 25(0.88)^t$.

3. A population P of deer, grows according to the formula $p = 25(1.07)^t$, where t is measured in years. The population of deer is increasing by:

(a) 0.07% per year
(b) 0.07 deer per year
(c) 7% per year
(d) 7 deer per year
(e) 1.07% per year
(f) 1.07 deer per year
(g) 25% per year
(h) 25 deer per year

ANSWER:

(c) Since the formula given is an exponential formula, the population is increasing by a constant percent per year. Since the base is $1.07 = 1 + 0.07$, the population is increasing by 7% per year.

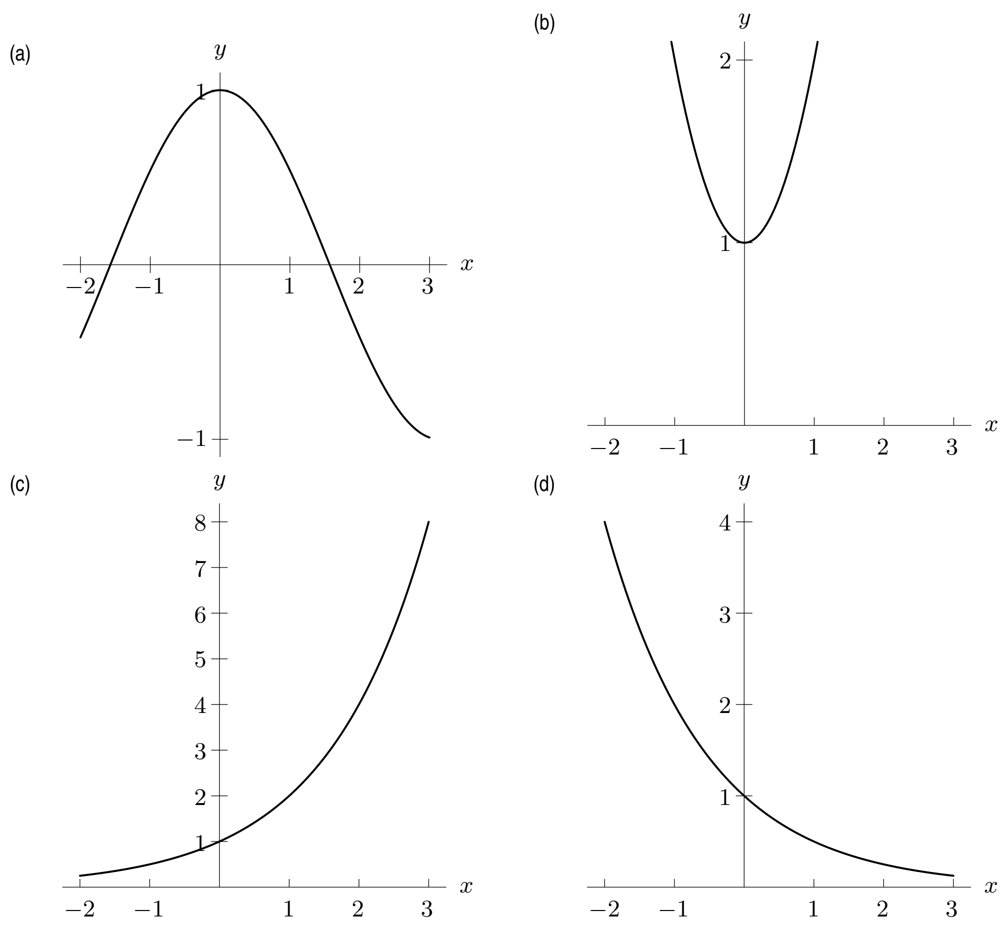
4. Sales at a company are changing according to the formula $S = 1000(0.82)^t$, where S is sales in thousands of dollars and t is measured in years. Sales at this company are:

   (a) Increasing by 82% per year
   (b) Increasing by 82 thousand dollars per year
   (c) Decreasing by 82% per year
   (d) Decreasing by 82 thousand dollars per year
   (e) Increasing by 18% per year
   (f) Increasing by 18 thousand dollars per year
   (g) Decreasing by 18% per year
   (h) Decreasing by 18 thousand dollars per year

   ANSWER:

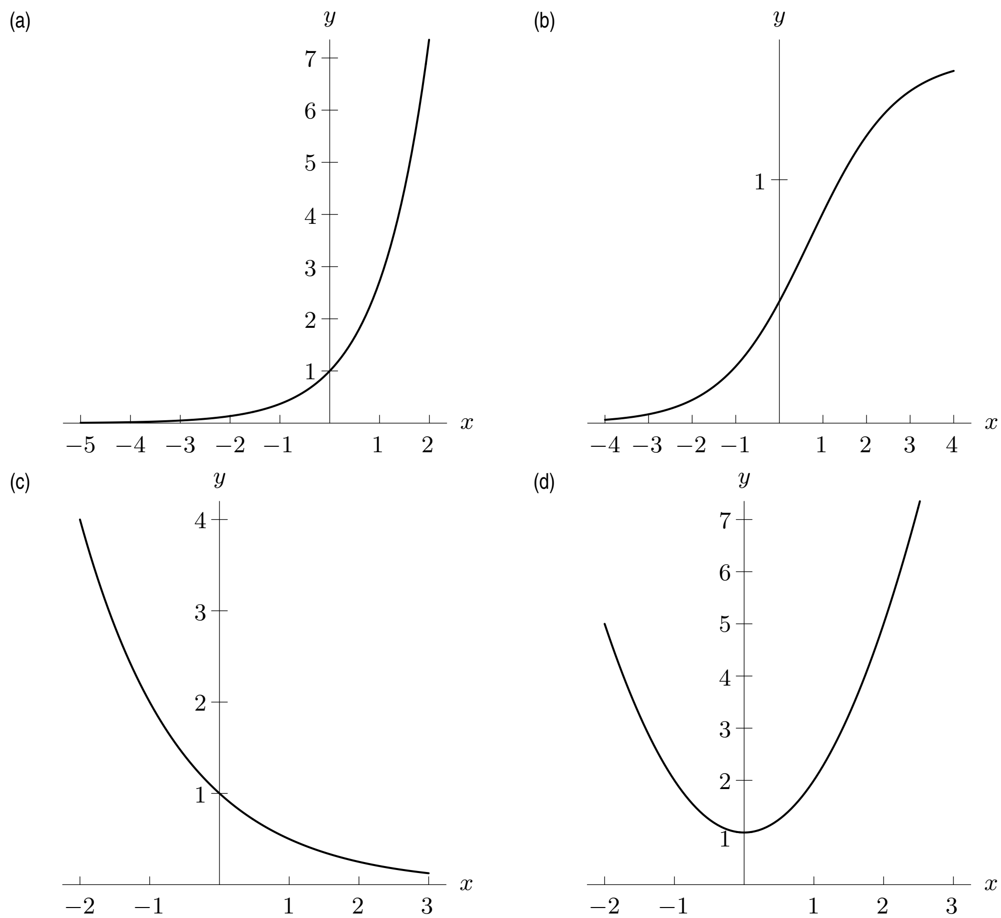
   (g) Since the formula given is an exponential formula, the population is changing by a constant percent per year. Since the base is less than one, the sales are decreasing. The base is $0.82 = 1 - 0.18$ so sales are decreasing by 18% per year.

5. Which of the graphs is that of $y = 2^x$?



ANSWER:
(c)
COMMENT:
Have students explain why each of the other choices is not appropriate.

6. Which of the graphs is that of $y = 2^{-x}$?



ANSWER:
(c)
COMMENT:
Have students explain why each of the other choices is not appropriate.

7. Every exponential function has a vertical intercept.

(a) True
(b) False

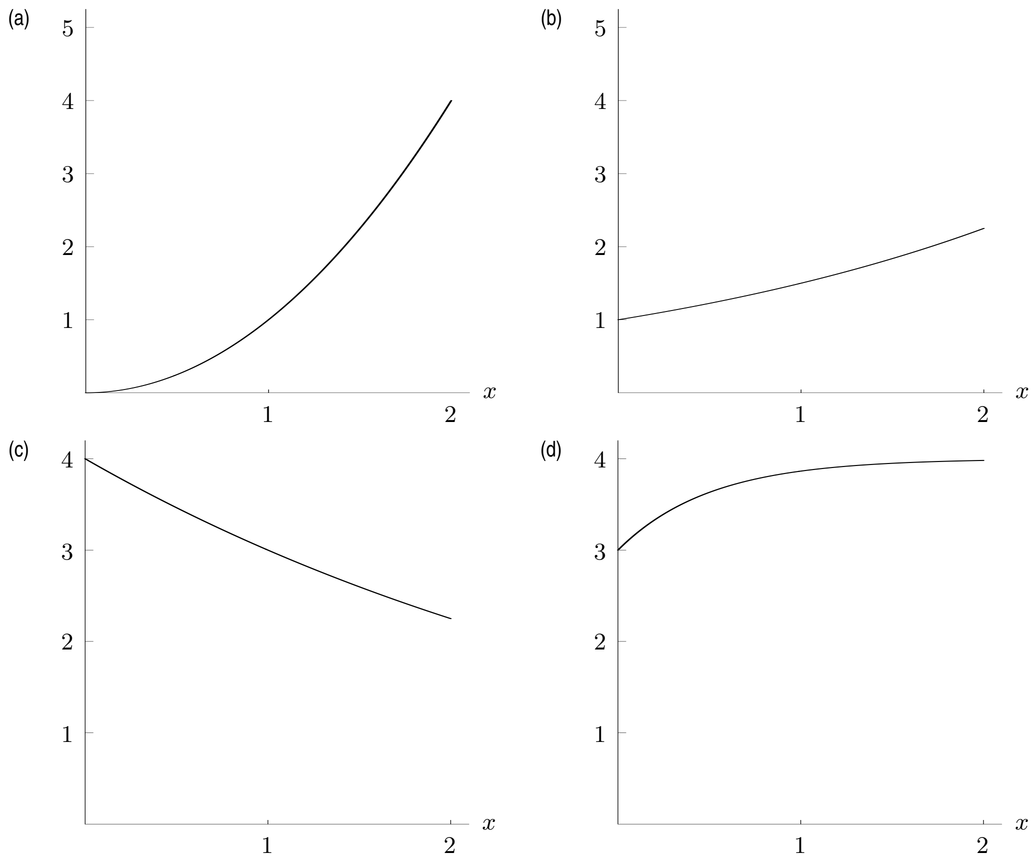
ANSWER:
(a). Exponential functions have the form $P_0 a^t$, where $a > 0$, but $a \neq 1$. The vertical intercept is P_0.
COMMENT:
Show this graphically—include positive and negative values of P_0 and values of a which are between 0 and 1 as well as values of a which are greater than 1.

8. Every exponential function has a horizontal intercept.

(a) True
(b) False

ANSWER:
(b). For example, $y = 2^x$ has no horizontal intercept.
COMMENT:
Exponential functions are of the form $y = P_0 a^t$ with $0 < a < 1$ or $a > 1$. Point out that an exponential function does not have a horizontal intercept unless $P_0 = 0$.

9. Which of the following graphs is that of $y = ab^x$ if $b > 1$?



ANSWER:
(b). If $b > 1$, we need a graph that is increasing, concave up, and does not go through the origin.
COMMENT:
Here it is worth having the students point out specifically why the other choices do not work.
Follow-up Questions. What is the value of a? How does the graph change if $a < 0$?

10. Let $f(x) = ab^x$, $b > 0$. Then $\dfrac{f(x+h)}{f(x)} =$

(a) b^h
(b) h
(c) $b^{x+h} - b^x$
(d) a

ANSWER:
(a), since $f(x+h) = ab^{x+h} = ab^x b^h = f(x)b^h$.
COMMENT:
This fact, in a different format, is used to find the derivative of b^x. This introduces the algebraic manipulations required for the definition of the derivative later in the text.

11. List at least five properties of exponential functions of the form $y(x) = ab^x$ for $a > 0$ and either $0 < b < 1$ or $b > 1$.
ANSWER:
(a) Domain: all real numbers
(b) Range: $y > 0$
(c) Horizontal asymptote: $y = 0$
(d) $y(0) = a$
(e) Always concave up
(f) Always increasing for $b > 1$, and always decreasing for $0 < b < 1$
(g) No vertical asymptote
COMMENT:
You can have your students list more properties. Discuss what happens if $a < 0$.

ConcepTests for Section 1.6

1. The net worth of a company is 20 million dollars and is increasing at a continuous rate of 5.3% per year. A possible formula for the net worth W in millions of dollars as a function of year t is:
(a) $W = 20 + 5.3t$
(b) $W = 20 + 0.053t$
(c) $W = 20(1.053)^t$
(d) $W = 20(1.53)^t$
(e) $W = 20(0.053)^t$
(f) $W = 20(0.53)^t$
(g) $W = 20e^{1.053t}$
(h) $W = 20e^{1.53t}$
(i) $W = 20e^{0.053t}$
(j) $W = 20e^{0.53t}$
ANSWER:
(i) Since the net worth is increasing by a constant percent each time unit, the function is exponential. Since the rate is continuous, we use the e^{kt} form for the exponential function. Since the rate is 5.3%, we have $k = 0.053$.

2. The number of acres of old growth forest in a park, N, initially 500 acres, is decreasing at a continuous rate of 15% per year. A possible formula for N as a function of t, measured in years, is:
(a) $N = 500(1.15)^t$
(b) $N = 500(0.85)^t$
(c) $N = 500(0.15)^t$
(d) $N = 500(-0.15)^t$
(e) $N = 500e^{1.15t}$
(f) $N = 500e^{0.85t}$
(g) $N = 500e^{0.15t}$
(h) $N = 500e^{-0.15t}$
(i) $N = 500 - 15t$
(j) $N = 500 - 0.15t$
ANSWER:
(h) Since the number of acres is decreasing by a constant percent each time unit, the function is exponential. Since the rate is continuous, we use the e^{kt} form for the exponential function. Since the number is decreasing, the value of k is negative.

3. Money in a bank grows according to the formula $B = 1000e^{0.04t}$, where B is in dollars and t is measured in years. The money is increasing by:

(a) 0.04% per year, compounded continuously
(b) 0.04% per year, compounded annually
(c) 0.04 dollars per year
(d) 4% per year, compounded continuously
(e) 4% per year, compounded annually
(f) 4 dollars per year

ANSWER:

(d) Since the formula given is exponential, the amount of money is increasing by a constant percent per year. Since the base is e, the percent is a continuous rate and the interest is compounded continuously. Since 0.04 is 4%, the answer is (d).

4. The quantity of a drug in the body is changing according to the formula $C = 25e^{-0.08t}$, where C is quantity in the body in mg and t is measured in hours. The quantity in the body is:

(a) Increasing at a continuous rate of 8% per hour
(b) Increasing by 8 mg per hour
(c) Decreasing at a continuous rate of 8% per hour
(d) Decreasing by 8 mg per hour
(e) Increasing at a continuous rate of 0.08% per hour
(f) Increasing by 0.08 mg per hour
(g) Decreasing at a continuous rate of 0.08% per hour
(h) Decreasing by 0.08 mg per hour

ANSWER:

(c) Since the formula given is exponential, the quantity of the drug is changing by a constant percent per hour. Since the exponent is negative, the quantity is decreasing. Since the exponent is $-0.08t$, the quantity is decreasing at a continuous rate of 8% per hour.

5. Converting the function $P = 100(1.07)^t$ to the form $P = P_0e^{kt}$ gives

(a) $P = 100e^{1.07t}$
(b) $P = 100e^{0.07t}$
(c) $P = 100e^{1.0677t}$
(d) $P = 100e^{0.0677t}$
(e) $P = 100e^{0.93t}$

ANSWER:

(d) The percent growth rate in the formula $P = 100(1.07)^t$ is 7% per unit of t. The equivalent continuous growth rate is close to, but not identical to, the percent growth rate of 7%, so a continuous growth rate of 6.77% is reasonable, leading us to the answer in (d). This analysis is enough to answer the question, but we can check our work by doing the calculation.

We have $P_0 = 100$ and we want to find k with $100(1.07)^t = 100(e^k)^t$, so we solve $1.07 = e^k$ and find $k = 0.0677$.

COMMENT:

Ask why the continuous growth rate is slightly smaller than 7%. Students should be able to answer this question without doing any calculations. Students should understand the e^{kt} form in the sense that we use $k = 0.0677$ not $k = 1.0677$, and students should understand that the values of $(1.07)^t$ and $e^{0.07t}$ are close but not identical.

6. Converting the function $P = 750e^{0.04t}$ to the form $P = P_0 a^t$ gives
   (a) $P = 750(1.04)^t$
   (b) $P = 750(0.04)^t$
   (c) $P = 750(1.0408)^t$
   (d) $P = 750(0.0408)^t$
   (e) $P = 750(0.96)^t$

   ANSWER:

   (c) The continuous percent growth rate in the formula $P = 750e^{0.04t}$ is 4% per unit of t. The equivalent non-continuous percent growth rate is close to, but not identical to, the continuous percent growth rate of 4%, so a growth rate of 4.08% is reasonable, leading us to the answer in (c). This analysis is enough to answer the question, but we can check our work by doing the calculation.

   We have $P_0 = 750$ and we want to find a with $750(e^{0.04})^t = 750(a)^t$, so we solve $a = e^{0.04} = 1.0408$.

   COMMENT:

   Ask why the non-continuous growth rate is larger than 4%. Students should be able to answer this question without doing any calculations. Students should understand the two forms of an exponential function, and should understand that values of $e^{0.04t}$ and $(1.04)^t$ are close but not identical.

7. The solution to $100 = 50e^t$ is:
   (a) $t = \ln(2)$
   (b) $t = \dfrac{\ln(100)}{\ln(50)}$
   (c) $t = \dfrac{\ln(100)}{50}$
   (d) $t = 100e^{50}$

   ANSWER:

   (a) We first divide both sides of the equation by 50 to obtain the equivalent equation $2 = e^t$. To solve for t, we then take the natural logarithm of both sides to see that $t = \ln(2)$.

8. The solution to $200 = 30e^{0.15t}$ is:
   (a) $t = \dfrac{\ln(200/30)}{\ln(0.15)}$
   (b) $t = \dfrac{\ln(200/30)}{0.15}$
   (c) $t = \ln\left(\dfrac{200}{30 \cdot (0.15)}\right)$
   (d) $t = \dfrac{200}{30}\ln(0.15)$

   ANSWER:

   (b) We divide both sides by 30 and then take the natural logarithm of both sides:

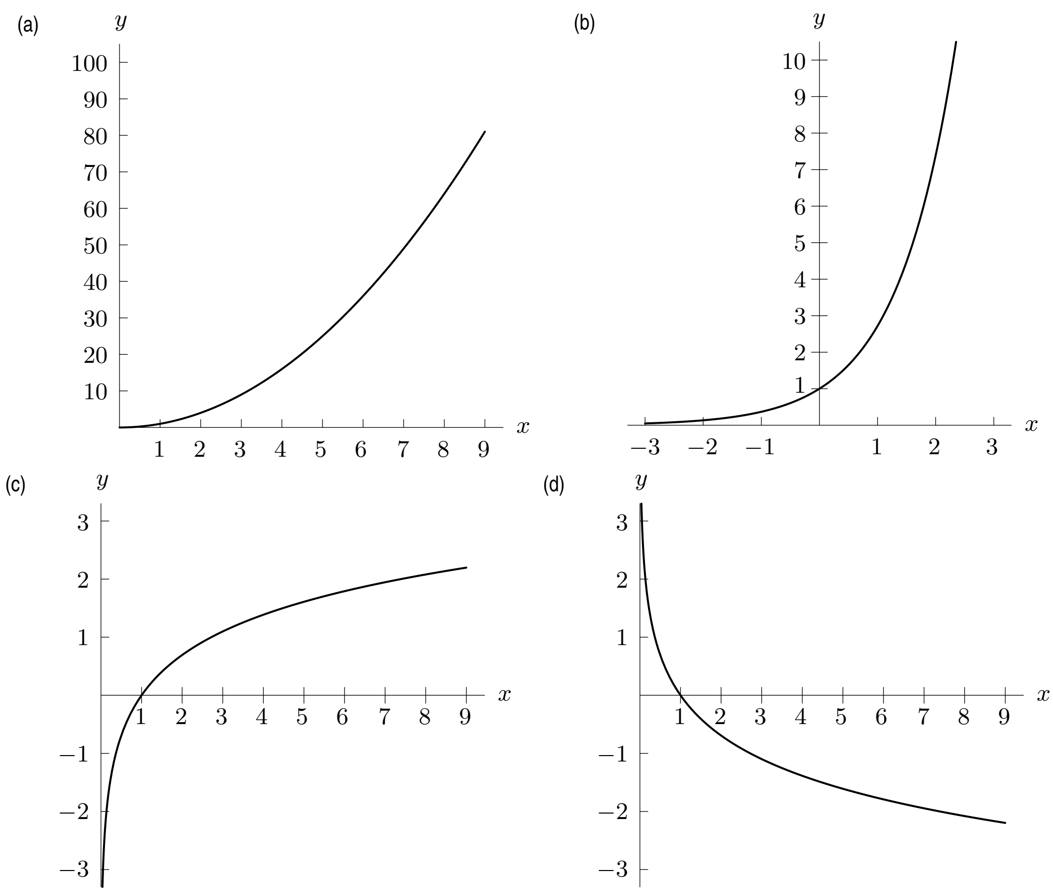
$$\begin{aligned} \frac{200}{30} &= e^{0.15t} \\ \ln\left(\frac{200}{30}\right) &= 0.15t \\ \frac{\ln(200/30)}{0.15} &= t. \end{aligned}$$

9. Each of the following equations has one variable in it. For which equations would you use logarithms to solve the equation analytically for the variable?
   I. $200 = 157(1.05)^t$
   II. $200 = 157(1 + r)^{20}$
   III. $F = 157e^{1.2}$
   IV. $200 = 157e^{k \cdot 4}$

   (a) I only
   (b) II only
   (c) III only
   (d) IV only
   (e) I and II only
   (f) II and III only
   (g) II and IV only
   (h) I and IV only
   (i) All of them
   (j) None of them

   ANSWER:
   (h) We use logarithms to solve for a variable in an exponent. Since the variable is in the exponent in equations I and IV, and not in equations II and III, the answer is (h).

10. Which is a graph of $y = \ln x$?



ANSWER:
(c). $y = \ln x$ is an increasing function passing through the point $(1, 0)$.
COMMENT:
You could discuss which properties each of the remaining graphs possess that made the student conclude that it was not an appropriate choice.
Follow-up Question. Find possible formulas for the remaining graphs.
Answer. (a) $y = x^2$, (b) $y = e^x$, and (d) $y = -\ln x$.

11. Which of the following functions are increasing and concave up?

(a) 3^{-x}
(b) 3^x
(c) $\ln x$
(d) $-\ln x$

ANSWER:
(b). Note that (a) and (d) are decreasing, and (c) is concave down.
COMMENT:
You could also ask about asymptotes (horizontal and vertical) and intercepts for all four functions.

12. Without caculating the following quantities, use the properties of logarithms to decide which of them is largest.

(a) $\ln(30) - \ln(2)$
(b) $2\ln 4$
(c) $\ln 3 + \ln 4$
(d) $\dfrac{\ln 4}{\ln 2}$

ANSWER:
(b). $\ln(30) - \ln(2) = \ln(15)$, $2\ln(4) = \ln(16)$, $\ln(3) + \ln(4) = \ln(12)$, and $\dfrac{\ln(4)}{\ln(2)} = \dfrac{\ln(2^2)}{\ln 2} = \dfrac{2\ln 2}{\ln 2} = 2 = \ln(e^2)$. Since $e^2 < 9$ and $\ln x$ is an increasing function, $\ln(16)$ is the largest number.
COMMENT:
Point out that using the rules of logarithms enables us to compare exact values. If the comparison were made using a computer or calculator, we would likely be comparing approximate values.

13. The graph of a logarithmic function has a horizontal asymptote.

(a) True
(b) False

ANSWER:
(b). The range of logarithmic functions consists of all real numbers.
COMMENT:
You could also ask about vertical asymptotes.

14. $\ln\left(\dfrac{M-N}{M+N}\right) =$

(a) $2\ln M$
(b) $2\ln N$
(c) $-2\ln N$
(d) $\ln(M-N) - \ln(M+N)$

ANSWER:
(d)
COMMENT:
Follow-up Question. What is the value of $\ln\left(\dfrac{M^2-N^2}{M-N}\right)$?
Answer. $\ln(M+N)$.

15. Which of the following do *not* have a horizontal asymptote?

(a) $y = \ln x$
(b) $y = \dfrac{1}{x}$
(c) $y = 5^x$
(d) $y = x^{1/3}$

ANSWER:
(a) and (d). The range of $\ln x$ and $x^{1/3}$ is all real numbers.
COMMENT:
Follow-up Question. Which of the above functions does *not* have any asymptotes?
Answer. (d). The domain and range of $y = x^{1/3}$ is all real numbers.

16. Solve for x if $8y = 3e^x$.
   (a) $x = \ln 8 + \ln 3 + \ln y$
   (b) $x = \ln 3 - \ln 8 + \ln y$
   (c) $x = \ln 8 + \ln y - \ln 3$
   (d) $x = \ln 3 - \ln 8 - \ln y$

   ANSWER:
   (c). If $8y = 3e^x$, then $\frac{8y}{3} = e^x$ and $\ln\left(\frac{8y}{3}\right) = x$, so $x = \ln 8 + \ln y - \ln 3$.
   COMMENT:
   This is a good place to point out the many ways answers can be expressed using logarithms.

17. Solve for x if $y = e + 2^x$.
   (a) $x = \frac{\ln y - 1}{\ln 2}$
   (b) $x = \frac{\ln(y-1)}{\ln 2}$
   (c) $x = \frac{\ln y}{\ln 2} - 1$
   (d) $x = \frac{\ln(y-e)}{\ln 2}$

   ANSWER:
   (d). If $y = e + 2^x$, $y - e = 2^x$ and $\ln(y-e) = x\ln 2$. This gives $x = \frac{\ln(y-e)}{\ln 2}$.
   COMMENT:
   You could ask what errors could have been made in obtaining the other choices.

18. For what value of x is $3 \cdot 3^{-x} + 4 = 16 - 3^{-x}$?
   ANSWER:
   If $3 \cdot 3^{-x} + 4 = 16 - 3^{-x}$, then $(3+1)3^{-x} = 16 - 4 = 12$. Division gives $3^{-x} = 3$, so $-x = 1$ and $x = -1$.
   COMMENT:
   Students may try to take logarithms of both sides of the original equation.

19. *"During 1988, Nicaragua's inflation rate averaged 1.3% a day."* Which formula represents the preceding statement? Assume t is measured in days.
   (a) $I = I_0 e^{0.013t}$
   (b) $I = I_0(1.013)^t$
   (c) $I = I_0(1.013)t$
   (d) $I = I_0(1.3)^t$

   ANSWER:
   (b)
   COMMENT:
   Follow-up Question. What happens if the statement is changed to *"During 1988, Nicaragua's inflation rate grew continuously at a rate of* 1.3% *each day."*?

ConcepTests for Section 1.7

1. A quantity starts at 30 mg and decays exponentially to 20 mg 3 hours later. Which of the following equations has a solution which gives the continuous decay rate?
   (a) $30 = 20e^{k\cdot 3}$
   (b) $20 = 30e^{k\cdot 3}$
   (c) $30/3 = 20e^k$
   (d) $20 = (3k)\cdot 30$
   (e) $20/3 = 30e^k$

   ANSWER:
   (b) In the formula $P = P_0 e^{kt}$, we have $P_0 = 30$. When we substitute the additional information that when $t = 3$ we have $P = 20$, we obtain the equation given in (b).

2. A sum of \$1000 is deposited in a bank account that pays 5% interest compounded continuously. Which of the following equations has a solution which tells you how long it takes for the money to double?
   (a) $P = 1000e^{0.05\cdot 2}$
   (b) $1000 = P_0e^{0.05\cdot 2}$
   (c) $1000 = 2000e^{0.05\cdot t}$
   (d) $2000 = 1000e^{0.05\cdot t}$

   ANSWER:

   (d) In the formula $P = P_0e^{kt}$, we have $P_0 = 1000$ and $k = 0.05$. When we solving for the value of t when $P = 2 \cdot 1000 = 2000$.

3. The half-life of caffeine is 4 hours. A person drinks a cup of coffee with 100 mg of caffeine and we wish to determine the number of hours until the amount of caffeine in the body is down to 10 mg. Which of the following equations has a solution which gives the continuous decay rate for caffeine in the body?
   (a) $10 = 100e^{k\cdot 4}$
   (b) $100 = 10e^{k\cdot 4}$
   (c) $50 = 100e^{k\cdot 4}$
   (d) $10 = (100/4)e^{k}$
   (e) $P = (100/10)e^{4}$

   ANSWER:

   (c) In the formula $P = P_0e^{kt}$, we have $P_0 = 100$. The information we are given about the rate of decay is that the half-life is 4 hours, so we know that when $t = 4$ we have $P = (1/2) \cdot 100 = 50$. Substituting this information gives us the equation in (c). Notice that the question in the problem (determine the number of hours until the amount of caffeine is down to 10 mg) is not relevant until we have found the continuous decay rate k.

4. Doubling times for three different interest rates, compounded continuously, are: 4.6 years, 6.9 years, 13.9 years. The interest rates are 5%, 10%, and 15%. Match the interest rates to the doubling times.
   (a) 5%: 4.6 years; 10%: 6.9 years; 15%: 13.9 years
   (b) 5%: 4.6 years; 10%: 13.9 years; 15%: 6.9 years
   (c) 5%: 6.9 years; 10%: 4.6 years; 15%: 13.9 years
   (d) 5%: 6.9 years; 10%: 13.9 years; 15%: 4.6 years
   (e) 5%: 13.9 years; 10%: 4.6 years; 15%: 6.9 years
   (f) 5%: 13.9 years; 10%: 6.9 years; 15%: 4.6 years

   ANSWER:

   (f) The larger the interest rate, the faster the bank account grows, and the shorter the time until the money in the bank account doubles.

5. Half-lives for three different continuous decay rates are: 2.3 hours, 3.5 hours, 6.9 hours. The decay rates, with time measured in hours, are 10%, 20%, and 30%. Match the decay rates to the half-lives.
   (a) 10%: 2.3 years; 20%: 3.5 years; 30%: 6.9 years
   (b) 10%: 2.3 years; 20%: 6.9 years; 30%: 3.5 years
   (c) 10%: 3.5 years; 20%: 2.3 years; 30%: 6.9 years
   (d) 10%: 3.5 years; 20%: 6.9 years; 30%: 2.3 years
   (e) 10%: 6.9 years; 20%: 2.3 years; 30%: 3.5 years
   (f) 10%: 6.9 years; 20%: 3.5 years; 30%: 2.3 years

   ANSWER:

   (f) The decay rates are all negative rates, and the more negative the rate, the faster the quantity decays and the shorter the half-life.

6. Future values of \$2000 at three different times in the future, assuming 8% interest, compounded continuously, are: \$2984, \$4451, \$6640. The times are 5 years in the future, 10 years in the future, and 15 years in the future. Match the times to the future values.

(a) 5 years: \$2984; 10 years: \$4451; 15 years: \$6640
(b) 5 years: \$2984; 10 years: \$6640; 15 years: \$4451
(c) 5 years: \$4451; 10 years: \$2984; 15 years: \$6640
(d) 5 years: \$4451; 10 years: \$6640; 15 years: \$2984
(e) 5 years: \$6640; 10 years: \$2984; 15 years: \$4451
(f) 5 years: \$6640; 10 years: \$4451; 15 years: \$2984

ANSWER:

(a) Future values gives the balance in the account at the end of the time period if the money is deposited in the account and left there to earn interest. The longer the money is in the account, the more interest is earns, so the future value increases as the time period increases.

7. A \$5000 payment is to be made at some point in the future. The present value of the payment depends on the future time. Present values for three different times are: \$1506, \$2247, \$3352. The times are 5 years in the future, 10 years in the future, and 15 years in the future. We assume an 8% interest rate, compounded continuously. Match the future times to the present values.

(a) 5 years: \$1506; 10 years: \$2247; 15 years: \$3352
(b) 5 years: \$1506; 10 years: \$3352; 15 years: \$2247
(c) 5 years: \$2247; 10 years: \$1506; 15 years: \$3352
(d) 5 years: \$2247; 10 years: \$3352; 15 years: \$1506
(e) 5 years: \$3352; 10 years: \$1506; 15 years: \$2247
(f) 5 years: \$3352; 10 years: \$2247; 15 years: \$1506

ANSWER:

(f) The present value is less than the amount of the payment in all cases, since receiving money at some point in the future is not worth as much as receiving money today. The farther away the payment is, the less it is worth today. Thus, as the time of the payment extends farther into the future, the present value goes down.

8. You have your choice of receiving \$5000 now or receiving five equal payments of \$1000 each, paid once per year starting now. You can assume a 6% interest rate. Which is the best option financially (that is, which has the larger present value)?

(a) The payment of \$5000 now
(b) The five equal payments of \$1000, paid once per year
(c) The two options are equivalent

ANSWER:

(a) Since the amount of money you are receiving is the same (\$5000) in each case, it is clearly better to get the money up front since it will start earning interest immediately in that case.

COMMENT:

Students should not have to do any calculations for this problem. Money up front is always better, unless it is a smaller amount of money.

9. You are to receive three equal payments of \$2000 each, paid once per year starting now. You can assume a 5% interest rate, compounded continuously. The future value of the payments, on the day you receive the final payment, is:

(a) $6000e^{0.05\cdot 3}$
(b) $6000e^{0.05\cdot 2}$
(c) $2000e^{0.05\cdot 3} + 2000e^{0.05\cdot 2} + 2000e^{0.05\cdot 1}$
(d) $2000e^{0.05\cdot 2} + 2000e^{0.05\cdot 1} + 2000$

ANSWER:

(d) You receive the first payment now, the second payment in one year, and the third and final payment in two years, so we are finding the future value in two years. The payment you receive now could earn interest for two years, the payment you receive in one year could earn interest for one year, and the payment you receive in two years does not earn any interest, so the answer is (d).

10. Estimate the half-life for the exponential decay shown in Figure 1.9.

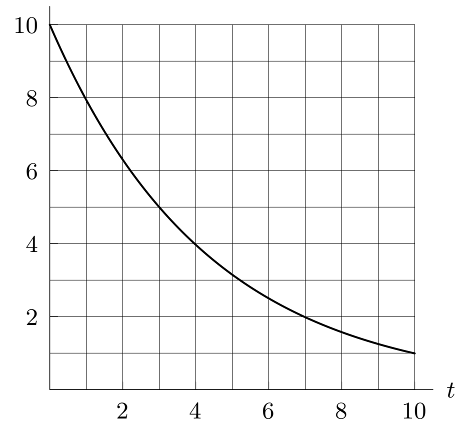


Figure 1.9

ANSWER:
3
COMMENT:
Depending on how much you enlarge the graph, some variation in answers should be allowed here.

11. Estimate the doubling time for the exponential growth shown in Figure 1.10.

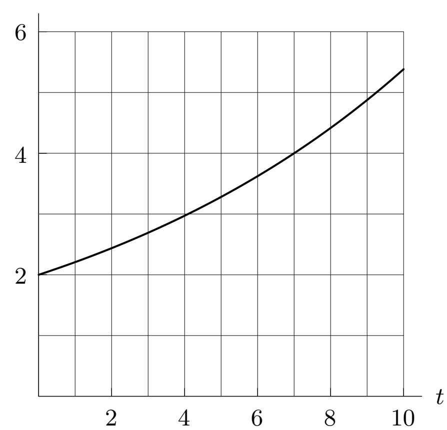


Figure 1.10

ANSWER:
7
COMMENT:
Depending on how much you enlarge the graph, some variation in answers should be allowed here.

ConcepTests for Section 1.8

For Problems 1–3, the graph in Figure 1.11 is that of $y = f(x)$. Use the graphs (I)–(IV) for the answers.

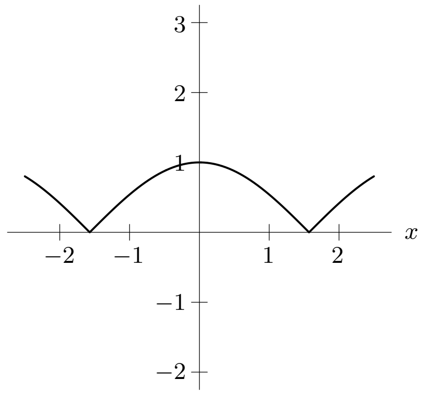
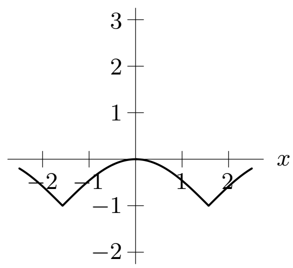
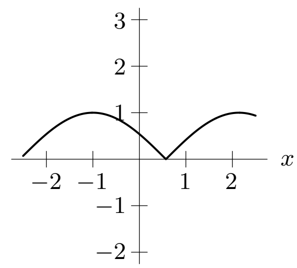


Figure 1.11

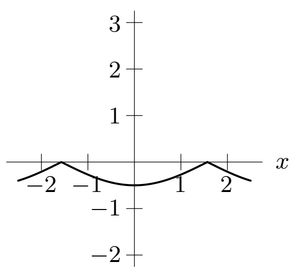
(I)



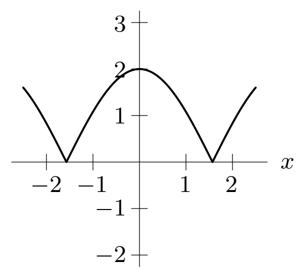
(II)



(III)



(IV)



1. Which could be a graph of $cf(x)$?
   ANSWER:
   (III) and (IV). In (III) the graph could be $-1/2f(x)$ and in (IV) the graph could be $2f(x)$.
   COMMENT:
   You could ask students to verbalize the relationship between $f(x)$ and $cf(x)$ for $c > 1, 0 < c < 1$, and $c < -1$.
2. Which could be a graph of $f(x) - k$?
   ANSWER:
   (I) could be $f(x) - 1$.
   COMMENT:
   You could ask students to verbalize the relationship between $f(x)$ and $f(x) - k$ for $k > 0$ and $k < 0$.
3. Which could be a graph of $f(x - h)$?
   ANSWER:
   (II) could be a graph of $f(x + 1)$.
   COMMENT:
   You could ask students to verbalize the relationship between $f(x)$ and $f(x - h)$ for $h > 0$ and $h < 0$.

For Problems 4–10, let f and g have values given in the table.

x	$f(x)$	$g(x)$
-2	1	-1
-1	0	1
0	-2	2
1	2	0
2	-1	-2

4. $f(g(1)) =$
   ANSWER:
   $g(1) = 0$, so $f(g(1)) = f(0) = -2$.
   COMMENT:
   You can also consider $f(g(x))$ for $x = -2$ and $x = 2$.
5. $f(g(0)) =$
   ANSWER:
   $g(0) = 2$, so $f(g(0)) = f(2) = -1$.
   COMMENT:
   You can also consider $f(g(x))$ for $x = -2$ and $x = 2$.
6. $f(g(-1)) =$
   ANSWER:
   $g(-1) = 1$, so $f(g(-1)) = f(1) = 2$.
   COMMENT:
   You can also consider $f(g(x))$ for $x = -2$ and $x = 2$.
7. If $f(g(x)) = 1$, then $x =$
   ANSWER:
   $f(-2) = 1$, and $g(2) = -2$, so $x = 2$.
   COMMENT:
   You can also consider $f(g(x)) = a$ for $a = -2, -1, 2$.
8. If $f(g(x)) = 0$, then $x =$
   ANSWER:
   $f(-1) = 0$, and $g(-2) = -1$, so $x = -2$.
   COMMENT:
   You can also consider $f(g(x)) = a$ for $a = -2, -1, 2$.
9. If $g(f(x)) = 2$, then $x =$
   ANSWER:
   $g(0) = 2$, and $f(-1) = 0$, so $x = -1$.
   COMMENT:
   You can also consider $g(f(x)) = a$ for $a = -1, 0, 1$.
10. If $g(f(x)) = -2$, then $x =$
    ANSWER:
    $g(2) = -2$, and $f(1) = 2$, so $x = 1$.
    COMMENT:
    You can also consider $g(f(x)) = a$ for $a = -1, 0, 1$.

For Problems 11–15, let the graphs of f and g be as shown in Figure 1.12. Estimate the values of the following composite functions to the nearest integer.

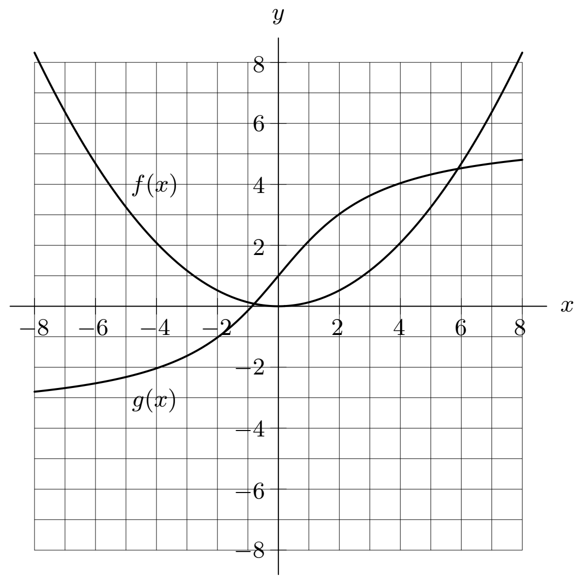


Figure 1.12

11. $g(f(0)) \approx$

ANSWER:

$f(0) \approx 0$, so $g(f(0)) \approx g(0) \approx 1$.

COMMENT:

You may want to point out that since $f(0) = 0$, this composition is similar to composing a function with the identity function.

12. $g(f(8)) \approx$

ANSWER:

$f(8) \approx 8$, so $g(f(8)) \approx g(8) \approx 5$.

COMMENT:

You may want to point out that since $f(8) = 8$, this composition is similar to composing a function with the identity function.

13. $g(f(3)) \approx$

ANSWER:

$f(3) \approx 1$, so $g(f(3)) \approx g(1) \approx 2$.

COMMENT:

When you are computing $g(f(a))$ from the graphs of g and f, it is not always necessary to compute $f(a)$. For example, when the horizontal and vertical scales are the same, you can measure the height of $f(a)$ with a straightedge. This distance placed on the x-axis is the new value from which to measure the height of g. The result will be $g(f(a))$.

14. $f(g(2)) \approx$

ANSWER:

$g(2) \approx 3$, so $f(g(2)) \approx f(3) \approx 1$.

COMMENT:

See the Comment for Problem 13.

15. $f(g(-1)) \approx$

ANSWER:

$g(-1) \approx 0$, so $f(g(-1)) \approx f(0) \approx 0$.

COMMENT:

See the Comment for Problem 13.

16. Given the graphs of the functions g and f in Figures 1.13 and 1.14, which of the following is a graph of $f(g(x))$?

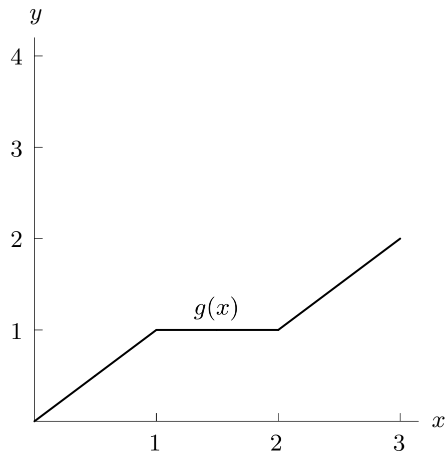


Figure 1.13

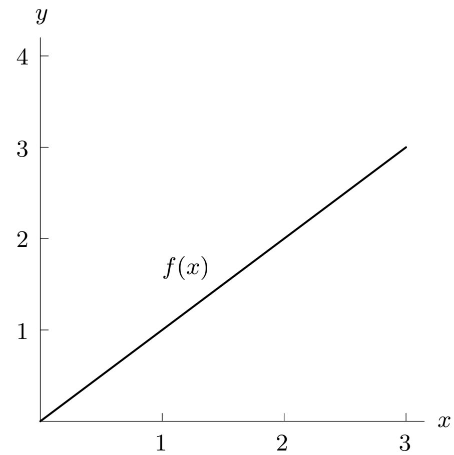
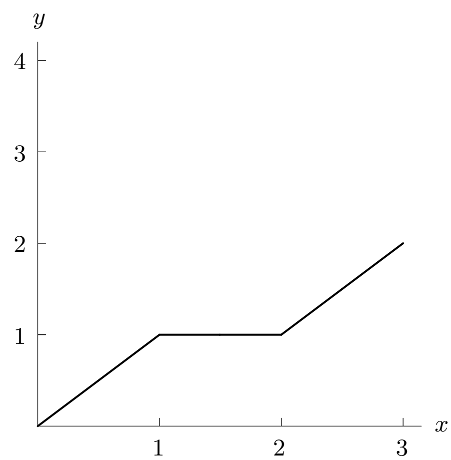
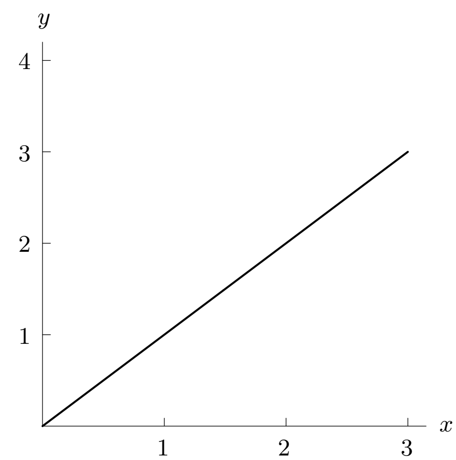


Figure 1.14

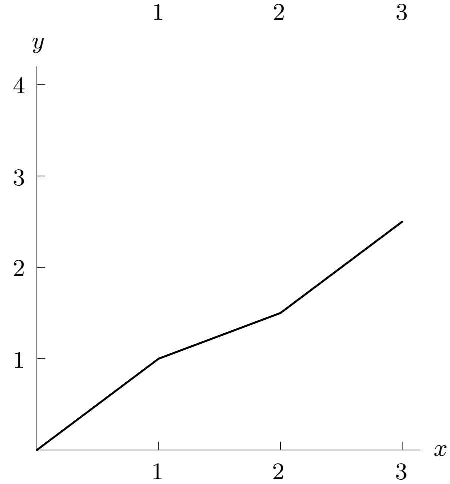
(a)



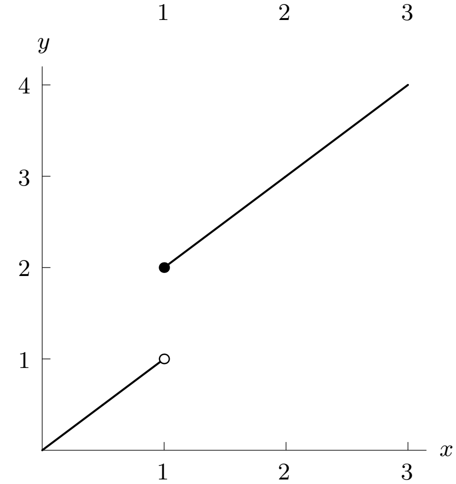
(b)



(c)



(d)



ANSWER:
(a). Because $f(x) = x$, we have $f(g(x)) = g(x)$.
COMMENT:
Follow-up Question. Which graph represents $g(f(x))$?

17. If $f(x) = \sqrt{x^2+1}$ and $g(x) = e^{x^2}$ then $f(g(x)) =$

(a) $e^{(x^2+1)}$
(b) $\sqrt{e^{2x^2}+1}$
(c) $e^{\sqrt{x^2+1}}$
(d) $\sqrt{e^{x^4}+1}$

ANSWER:
(b) Substituting $g(x)$ into $f(x)$ gives

$$f(g(x)) = \sqrt{\left(e^{x^2}\right)^2+1} = \sqrt{e^{2x^2}+1}.$$

COMMENT:
Students have trouble simplifying $\left(e^{x^2}\right)^2$. Next you could have them compute $g(f(x))$.

18. The graph in Figure 1.15 could be that of

(a) $y = \ln x + \dfrac{1}{2}$
(b) $y = \ln x - \dfrac{1}{2}$
(c) $y = \ln\left(x + \dfrac{1}{2}\right)$
(d) $y = \ln\left(x - \dfrac{1}{2}\right)$

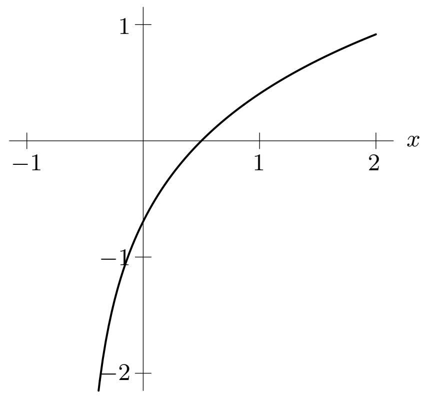


Figure 1.15

ANSWER:
(c). Note that (a) is the graph of $\ln x$ shifted up $\frac{1}{2}$, (b) is shifted down $\frac{1}{2}$, (d) is shifted to the right $\frac{1}{2}$.
COMMENT:
You could also distinguish the four graphs by their horizontal intercepts.

19. Which of the following functions have vertical asymptotes of $x = 3$?

(a) $y = \ln\left(\dfrac{x}{3}\right)$
(b) $y = \ln(x-3)$
(c) $y = \ln(x+3)$
(d) $y = 3\ln x$

ANSWER:
(b). Note that (a) and (d) have vertical asymptotes at $x = 0$, while (c) has one at $x = -3$, and (b) has one at $x = 3$, as desired.
COMMENT:
Follow-up Question. Do any of these functions have horizontal asymptotes? If so, what are they?
Answer. No, the range of these functions is all real numbers.

20. Which of the following functions are decreasing and concave up?
    (a) $-\ln(4+x)$
    (b) 3^{x-4}
    (c) 3^{4-x}
    (d) $\ln(4-x)$

    ANSWER:
    (a) and (c). Note that (b) is increasing and (d) is concave down.
    COMMENT:
    You could also ask about asymptotes (horizontal and vertical) and intercepts for all four functions.

ConcepTests for Section 1.9

1. The number, N, of species of reptiles found on an island is proportional to the fourth root of the area, A, of the island. Which of the following represents this statement?
    (a) $N = A^{1/4}$
    (b) $A = N^{1/4}$
    (c) $N = A^{-4}$
    (d) $A = N^{-4}$
    (e) $N = kA^{1/4}$
    (f) $A = kN^{1/4}$
    (g) $N = kA^{-4}$
    (h) $A = kN^{-4}$

    ANSWER:
    (e) The fourth root of A is $A^{1/4}$. "Proportional to" means a constant multiple of, so we have $N = k \cdot A^{1/4}$.

2. One quantity Q is inversely proportional to the cube of another quantity, W. Which of the following represents this statement?
    (a) $Q = W^{1/3}$
    (b) $Q = W^{-1/3}$
    (c) $Q = W^{-3}$
    (d) $Q = kW^{3}$
    (e) $Q = kW^{1/3}$
    (f) $Q = kW^{-1/3}$
    (g) $Q = kW^{-3}$
    (h) $Q = \frac{1}{k}W^{3}$
    (i) $Q = \frac{1}{k}W^{1/3}$
    (j) $Q = \frac{1}{k}W^{-1/3}$

    ANSWER:
    (g) We know that Q is inversely proportional to W^3, which means

$$Q = k\frac{1}{W^3} = kW^{-3}.$$

3. Rewrite each of the following as a power function in the form kx^n, or state that it is not a power function.

$$\frac{5}{x^3}; \quad \frac{\sqrt{x}}{3}; \quad \frac{7}{2^x}.$$

    (a) $(1/5)x^{-3}$; $3x^{1/2}$; Not power
    (b) $5x^{-3}$; $(1/3)x^{1/2}$; Not power
    (c) $5x^{1/3}$; $(1/3)x^{-2}$; Not power
    (d) $(1/5)x^{1/3}$; $3x^{-2}$; $7 \cdot 2^{-x}$
    (e) $5x^{-3}$; $(1/3)x^{1/2}$; $7 \cdot 2^{-x}$

    ANSWER:
    (b) The function $7/2^x$ is an exponential function, not a power function.

4. All linear functions are examples of direct proportionality.
   (a) True
   (b) False

   ANSWER:
   (b). Any linear function whose graph does not pass through the origin is not an example of direct proportionality.
   COMMENT:
   Students should try to find examples as well as counterexamples anytime a definition is introduced.

5. Which of the following graphs represent y as directly proportional to x?

(a)

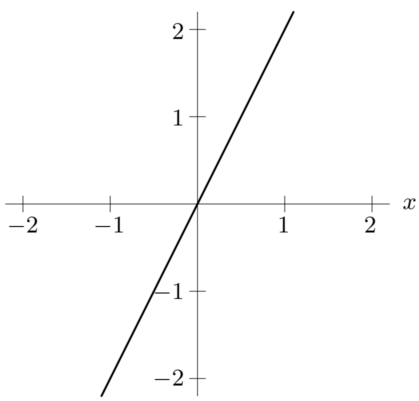


Figure 1.16

(b)

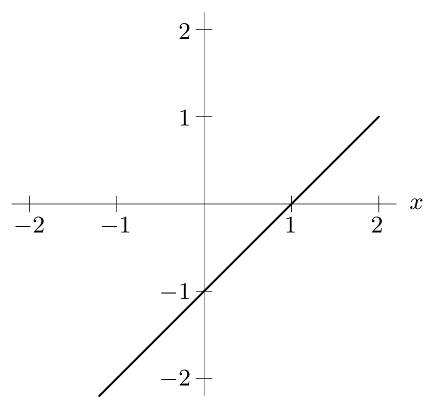


Figure 1.17

(c)

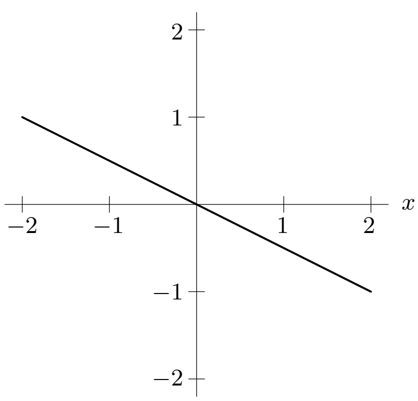


Figure 1.18

(d)

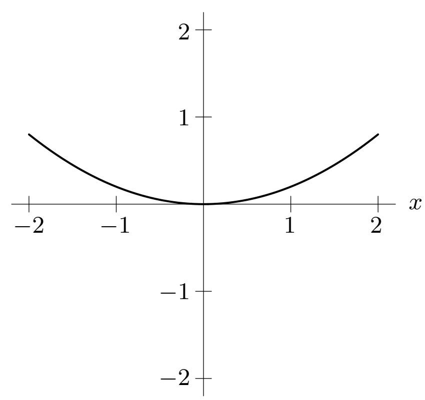
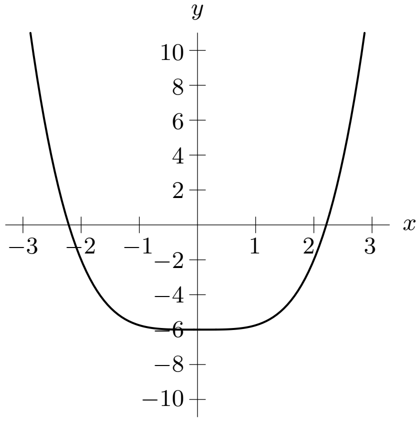


Figure 1.19

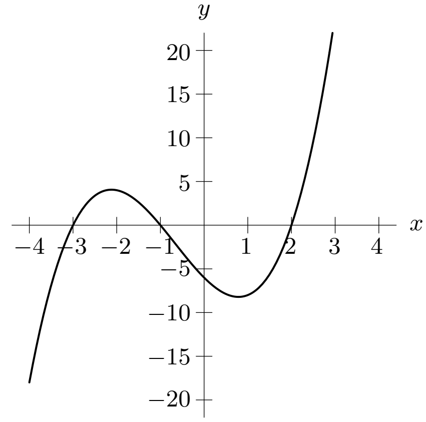
ANSWER:
(a) and (c). If y is directly proportional to x, then $y = kx$, where k is a constant. (a) and (c) are graphs of such equations.
COMMENT:
Note that graphs representing y as directly proportional to x are lines through the origin. Students should recognize the graphical properties of y being directly proportional to x. Notice that (d) could be a representation of y being directly proportional to some **even power** of x.

6. The equation $y = x^3 + 2x^2 - 5x - 6$ is represented by which graph?

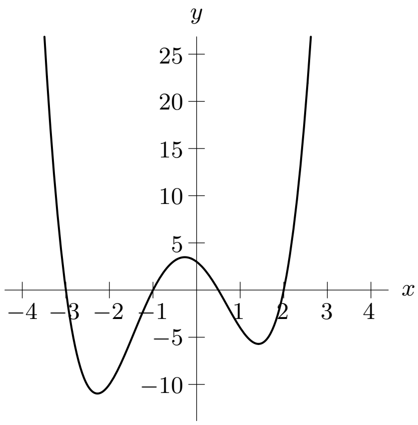
(a)



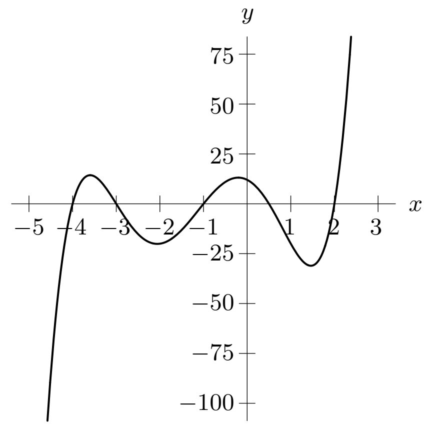
(b)



(c)



(d)



ANSWER:
(b). The graph will have a y-intercept of -6 and not be that of an even function.
COMMENT:
You may want to point out the various tools students can use to solve this problem, i.e. intercepts, even/odd, identifying the zeros, etc. You could also have students identify a property in each of the other choices that is inconsistent with the graph of $y = x^3 + 2x^2 - 5x - 6$.

7. The graph in Figure 1.20 is a representation of which function?

(a) $y = 3x + 2$ (b) $y = (x - 2)(x + 3)$ (c) $y = (x - 6)(x - 2)$
(d) $y = (x - 3)(x + 2)$

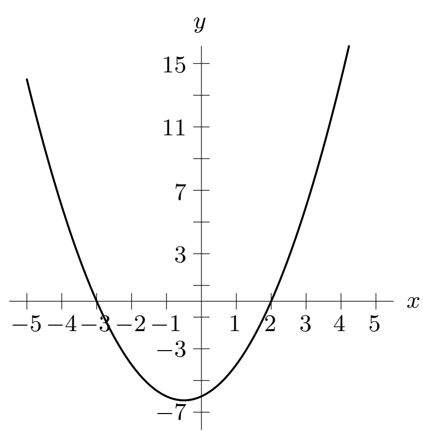
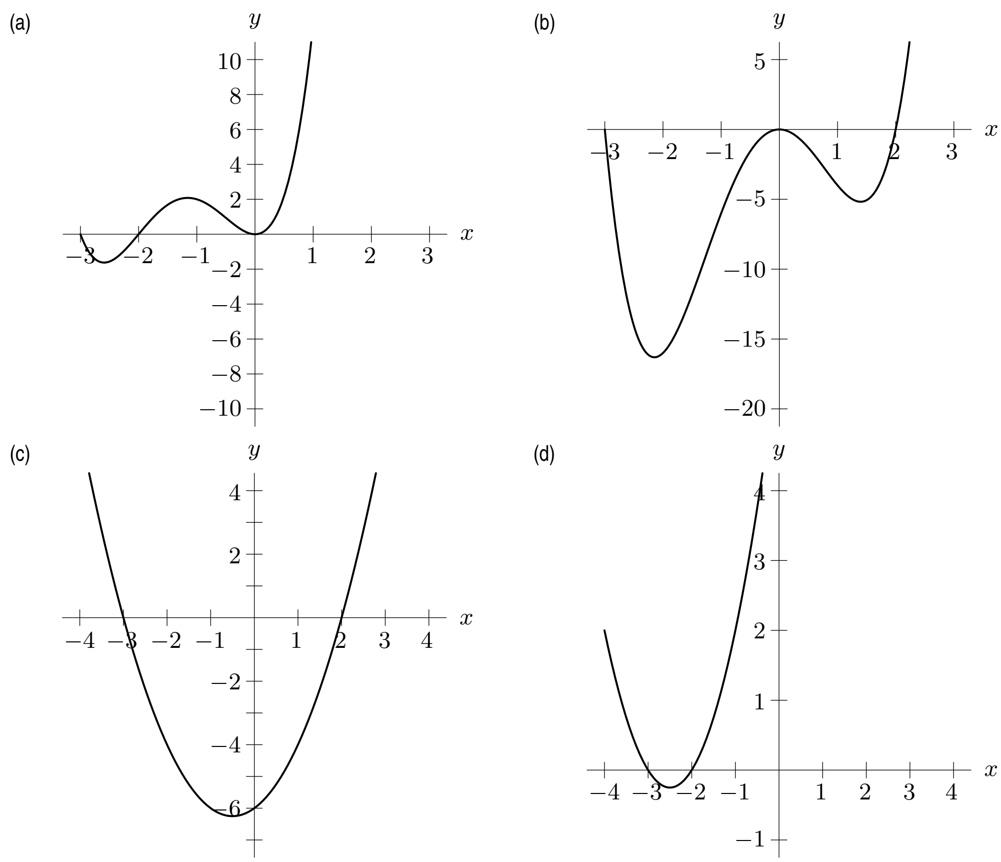


Figure 1.20

ANSWER:
(b). The graph is a parabola with x-intercepts of 2 and -3.
COMMENT:
You could ask students to describe the graphs of the other equations.

8. The equation $y = x^2 + 5x + 6$ is represented by which graph?



ANSWER:
(d). The graph will be a parabola with x-intercepts of -2 and -3.
COMMENT:
Have students identify a property in each of the other choices that is inconsistent with the graph of $y = x^2 + 5x + 6$.

9. Consider the function $f(x) = x^2 + 2x$. Give an equation of a line that intersects the graph of this function
   (a) Twice
   (b) Once
   (c) Never

ANSWER:
(a) There are many answers here. Any horizontal line of the form $y = a$, where $a > -1$.
(b) There are many answers here. One example is $y = -1$.
(c) There are many answers here. Any horizontal line of the form $y = a$, where $a < -1$.

COMMENT:
If you consider this question graphically, then have your students draw non-horizontal lines that meet the requirements for (a) and (b). This could be a way to introduce the idea of tangent lines.

ConcepTests for Section 1.10

1. A population oscillates between a high of 550 and a low of 450. The period is 2π and the population is at its peak at time $t = 0$. Give a formula that could represent the size of the population, P, as a function of time t.

   (a) $P = 550 \sin t$
   (b) $P = 550 \cos t$
   (c) $P = 50 \sin t + 500$
   (d) $P = 50 \cos t + 500$
   (e) $P = 50 + 500 \sin t$
   (f) $P = 50 + 500 \cos t$

   ANSWER:

   (d) Since the population is at its peak at $t = 0$, we use the cosine function. The population is centered at 500, so the cosine function is shifted vertically upward by 500. The population goes up and down by 50 from the central value, so the amplitude is 50. The function is $P = 50 \cos t + 500$.

   COMMENT:

   Have the students describe the oscillation given by the other functions. Which of these might reasonably represent the size of a population?

2. Sales of a product oscillate seasonally between a high of 1000 and a low of 200. The period is one year and time is measured in months since January 1. Sales are at their peak in the summer and at their lowest in the winter (on January 1). Give a formula that could represent sales, S, as a function of time t.

   (a) $S = 600 - 400 \cos t$
   (b) $S = -1000 + (\pi/6) \cos t$
   (c) $S = 400 + 600 \sin((\pi/6)t)$
   (d) $S = 400 - 600 \cos((\pi/6)t)$
   (e) $S = 600 - 400 \cos((\pi/6)t)$
   (f) $S = -\cos((\pi/6)t) + 1000$

   ANSWER:

   (e) Since the population is at its lowest point at $t = 0$, we use the negative cosine function. The population is centered at 600, so there is a vertical shift of 600. The population goes up and down by 400 from the center, so the amplitude is 400. The period is 12, so the coefficient of t in the argument is $2\pi/12 = \pi/6$. The function is $S = 600 - 400 \cos((\pi/6)t)$.

   COMMENT:

   Have the students describe the oscillation given by the other functions. Which of these might reasonably represent sales of a product?

3. The functions in Figure 1.21 have the form $y = A \sin x$. Which of the functions has the largest A? Assume the scale on the vertical axes is the same for each graph.

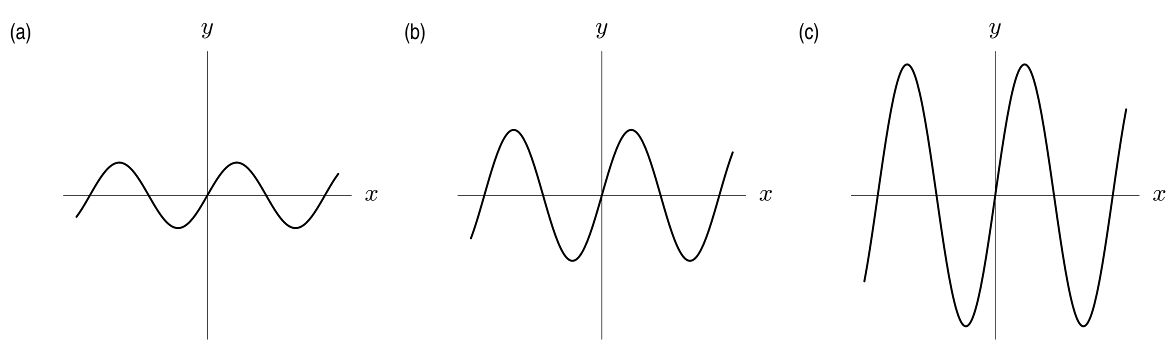


Figure 1.21

   ANSWER:

   (c) has the largest amplitude.

   COMMENT:

   Remind students that they only need to compare the graphs. They do not need any values on the axes labeled, but they need to know that the vertical scales are the same on all three graphs.

4. The graphs in Figure 1.22 have the form $y = \sin Bx$. Which of the functions has the largest B? Assume the scale on the horizontal axes is the same for each graph.

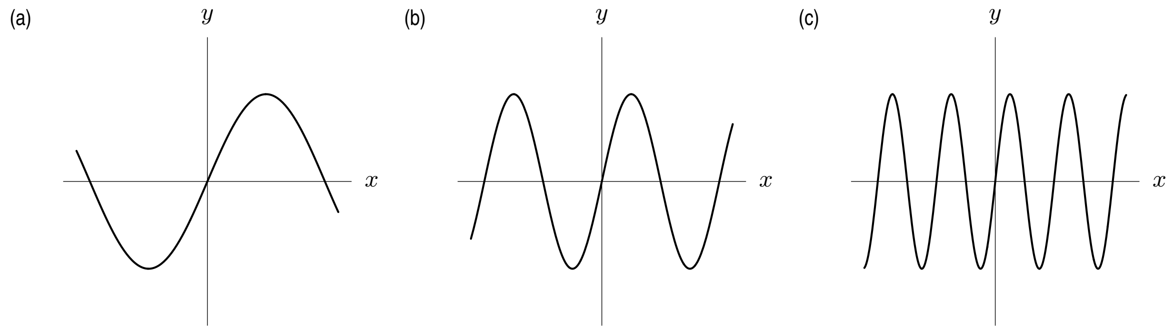


Figure 1.22

ANSWER:

(c). Since $\text{period} = 2\pi/|\text{B}|$, then the largest B corresponds to the smallest period.

COMMENT:

Remind the students that the horizontal scale is the same for all three graphs.

5. The amplitude and period of the graph of the periodic function in Figure 1.23 are

(a) Amplitude: 2. Period: 2.
(b) Amplitude: 2. Period: 3.
(c) Amplitude: 2. Period: $1/2$.
(d) Amplitude: 3. Period: 2.
(e) Amplitude: 3. Period: $1/2$.

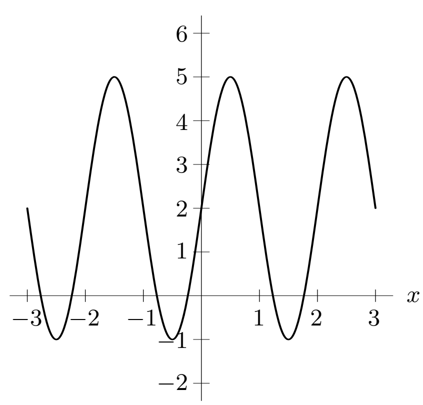


Figure 1.23

ANSWER:

(d). The maximum value of the function is 5, and the minimum value is -1, so the amplitude is $\dfrac{5-(-1)}{2} = 3$. The function repeats itself after 2 units, so the period is 2.

COMMENT:

Point out that the function is oscillating about the line $y = 2$. Have the students find a formula for the function shown in Figure 1.23.

6. The amplitude and period of the graph of the periodic function in Figure 1.24 are
   (a) Amplitude: 2. Period: 2.
   (b) Amplitude: 2. Period: 3.
   (c) Amplitude: 2. Period: $1/2$.
   (d) Amplitude: 3. Period: 2.
   (e) Amplitude: 3. Period: $1/2$.

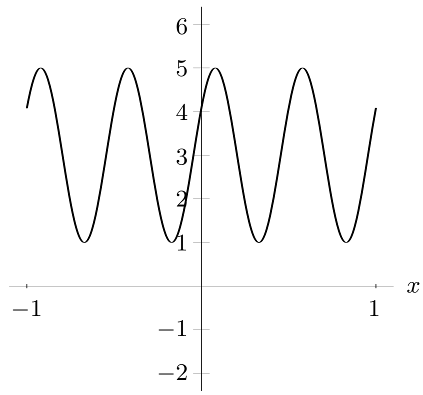


Figure 1.24

ANSWER:

(c). The maximum value of the function is 5 and the minimum is 1, so the amplitude is $\dfrac{5-1}{2} = 2$. The function repeats itself after $1/2$ unit, so the period is $1/2$.

COMMENT:

It is easiest to find the period using the extreme values of the function.

7. Which of the following could describe the graph in Figure 1.25.

(a) $y = 3\sin\left(\dfrac{x}{2} + \dfrac{\pi}{2}\right)$ (b) $y = 3\sin\left(2x + \dfrac{\pi}{2}\right)$ (c) $y = 3\cos(2x)$

(d) $y = 3\cos\left(\dfrac{x}{2}\right)$ (e) $y = 3\sin(2x)$ (f) $y = 3\sin\left(\dfrac{x}{2}\right)$

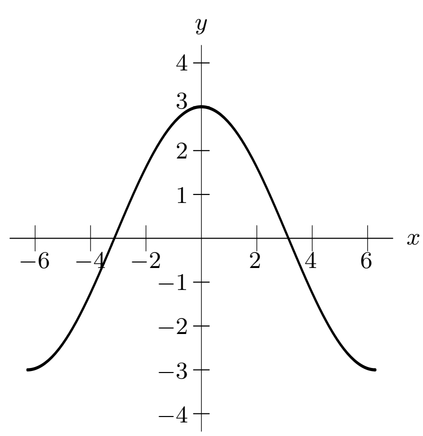


Figure 1.25

ANSWER:

(a) and (d). Note that (b), (c), and (e) have period π, with the rest having period 4π. Answer (f) has $y(0) = 0$. (a) and (d) could describe the graph.

COMMENT:

The fact that the same graph may have more than one analytic representation could be emphasized here.

8. Figure 1.26 shows the graph of which of the following functions?
   (a) $y = \cos(x + \pi/6)$
   (b) $y = \cos(x - \pi/6)$
   (c) $y = \sin(x - \pi/6)$
   (d) $y = \sin(x + \pi/6)$
   (e) None of these

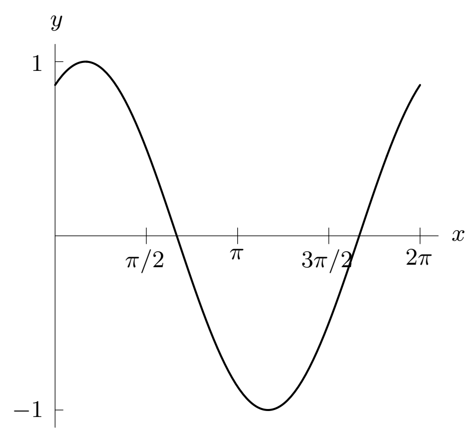


Figure 1.26

ANSWER:
(b)
COMMENT:
You could ask what the graphs of the other choices look like.

9. Figure 1.27 shows the graph of which of the following functions?

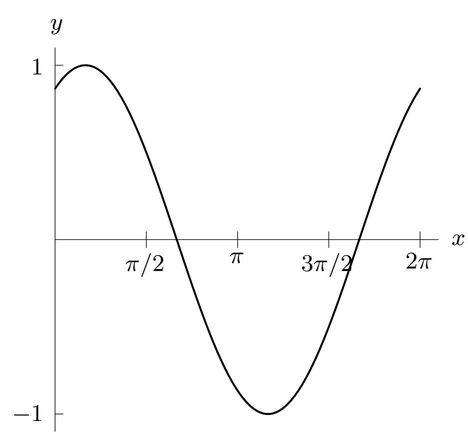


Figure 1.27

   (a) $y = \sin(x - \pi/3)$
   (b) $y = \cos(x - \pi/3)$
   (c) $y = \sin(x + \pi/3)$
   (d) $y = \cos(x + \pi/3)$
   (e) None of these

ANSWER:
(c)
COMMENT:
You could ask what the graphs of the other choices look like.

10. Given the graph of $y = \sin x$ in Figure 1.28, determine which of the graphs are those of $\sin(2x)$ and $\sin(3x)$?

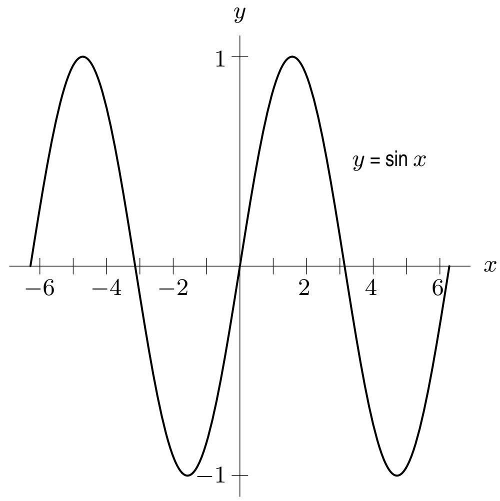
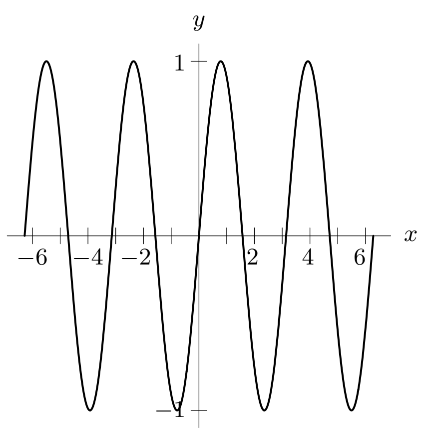
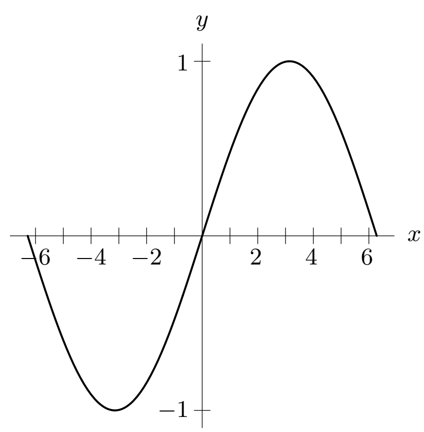


Figure 1.28

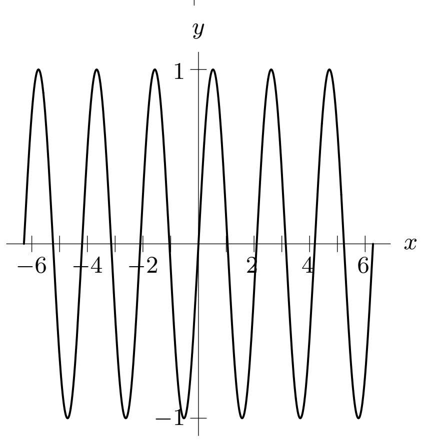
(I)



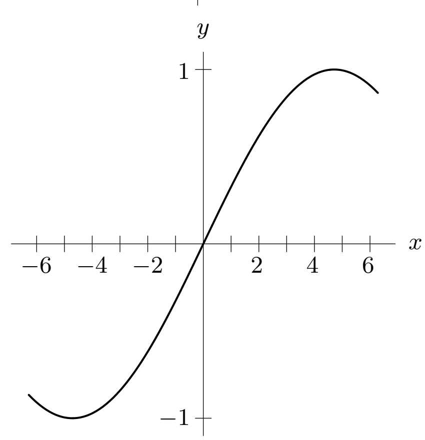
(II)



(III)



(IV)



(a) (I) $= \sin(2x)$ and (II) $= \sin(3x)$
(b) (I) $= \sin(2x)$ and (III) $= \sin(3x)$
(c) (II) $= \sin(2x)$ and (III) $= \sin(3x)$
(d) (II) $= \sin(2x)$ and (IV) $= \sin(3x)$
(e) (III) $= \sin(2x)$ and (IV) $= \sin(3x)$

ANSWER:

(b). Replacing x by $2x$ means the first positive x-intercept will be at $x = \pi/2$, so (I) is that of $y = \sin(2x)$. Similarly the first positive zero of $y = \sin(3x)$ is at $x = \pi/3$, so (III) is that of $y = \sin(3x)$.

COMMENT:

Have students also determine the equation for the graphs labeled (II) and (IV).

11. Consider the function $f(x) = 2\sin x$. Give an equation of a line that intersects the graph of this function

(a) Once (b) Never (c) An infinite number of times

ANSWER:

(a) There are many answers here. For example any vertical line will do.
(b) Any horizontal line of the form $y = n$ where $|n| > 2$.
(c) Any horizontal line of the form $y = n$ where $-2 \leq n \leq 2$.

COMMENT:

Follow-up Question. Draw a line that intersects the graph of this function

(a) Twice
(b) Three times
(c) Four times

You can also ask if it is possible for a line which intersects the curve at an intercept to intersect the curve an even number of times.

ConcepTests for Fitting Formulas to Data

1. A recent study used number of hours of TV per week watched as a child to predict adult body mass index (BMI). For a subset of this data, your calculator gives the regression line as:

$$Y = a + bx; \quad a = 18.2; \quad b = 0.48.$$

Which of the following is a valid interpretation of the slope of this line?

(a) If hours of TV watched per week as a child goes up by 18.2, adult BMI goes up by about 0.48.
(b) If adult BMI is 18.2 higher, then we expect hours of TV watched per week as a child to be 0.48 higher.
(c) Hours of TV watched per week as a child increases by an average of about 0.48 hours for every increase of one unit in adult BMI.
(d) Adult BMI increases by an average of about 0.48 for every additional hour of TV per week watched as a child.

ANSWER:

(d) We are using hours of TV to predict BMI, so the x-variable is hours of TV and the y-variable is BMI. The slope of the line is $b = 0.48$, so we have

$$\text{Slope} = 0.48 = \frac{0.48}{1} = \frac{\Delta y}{\Delta x} = \frac{\text{Change in BMI}}{\text{Change in TV hours}}.$$

The slope tells us that as TV hours per week increases by 1 unit, adult BMI increases on average by about 0.48, so the answer is (d).

COMMENT:

Have the students write out the regression line and interpret the vertical intercept as well.

2. A recent study found that number of hours of TV per week watched as a child is an effective variable to predict adult body mass index (BMI). Do you expect the correlation between these two variables to be positive, negative, or zero?

(a) Positive
(b) Negative
(c) Zero

ANSWER:

(a) Since the study found a relationship between the two variables, we do not expect the correlation to be zero. Since it seems reasonable that children who spend a great deal of time watching TV are more likely to be overweight as adults, we suspect that the correlation is positive.

COMMENT:

Ask the students what a negative correlation would mean in this situation. What would a zero correlation mean?

3. A study reports that, in a certain industry, there is strong evidence that as amount spent on health benefits goes down, absenteeism goes up. Which of the following values is the most reasonable estimate of the correlation between these two variables?

(a) 15.6
(b) 0.78
(c) 0
(d) -0.012
(e) -0.84
(f) -10.5
(g) -94.7

ANSWER:

(e) Correlation values are always between $+1$ and -1, so the values given in (a), (f), and (g) are impossible for a correlation. Since absenteeism goes up as health benefits go down, the correlation is negative. Since the evidence is strong, it is close to -1, so the best answer is (e).

COMMENT:

Ask the students what a positive correlation would mean for these two variables. What would a zero correlation mean?

4. A study reports that there does not appear to be a strong relationship between white blood cell count and heart rate. Which of the following values is the most reasonable estimate of the correlation between these two variables?

(a) 15.6
(b) 0.78
(c) -0.012
(d) -0.84
(e) -10.5
(f) -94.7

ANSWER:

(c) Correlation values are always between $+1$ and -1, so the values given in (a), (e), and (f) are impossible for a correlation. Correlations near zero indicate that there is no obvious linear relationship, so the best answer is (c).

COMMENT:

Ask the students what a positive correlation would mean for these two variables. What would a negative correlation mean?

5. For the data in Table 1.3, a calculator gives the regression line approximately as follows:

$$y = a + bx; \quad a = 90; \quad b = -0.5.$$

Give the predicted value for y when $x = 30$, and give the difference between the actual value when $x = 30$ and the predicted value when $x = 30$.

Table 1.3

x	10	20	30	40	50
y	85	80	77	70	64

(a) 75; 77
(b) 77; 2
(c) 75; 2
(d) 105; 28
(e) -15, 62
(f) 75; 15

ANSWER:

(c) The regression line is $y = 90 - 0.5x$ so the predicted value when $x = 30$ is $y = 90 - 0.5(30) = 90 - 15 = 75$. The actual y-value in the table corresponding to $x = 30$ is 77, so the difference between the actual value and the predicted value is $77 - 75 = 2$.

ConcepTests for Limits to Infinity and End Behavior

For Problems 1–5, as $x \to \infty$ which function dominates, (a) or (b)? (That is, which function is larger in the long run?)

1. **(a)** $0.1x^2$
(b) $10^{10}x$
ANSWER:
(a). Power functions with the power greater than one and with a positive coefficient grow faster than linear functions.
COMMENT:
You could ask about the behavior as $x \to -\infty$ as well.

2. **(a)** $0.25\sqrt{x}$
(b) $25{,}000x^{-3}$
ANSWER:
(a). Note that $0.25\sqrt{x}$ is an increasing function whereas $25{,}000x^{-3}$ is a decreasing function.
COMMENT:
One reason for such a question is to note that global behavior may not be determined by local behavior.

3. **(a)** $3 - 0.9^x$
(b) $\ln x$
ANSWER:
(b). Note that $3 - 0.9^x$ has a horizontal asymptote whereas the range of $\ln x$ is all real numbers.
COMMENT:
Students should realize that the graph the calculator displays can be misleading.

4. **(a)** x^3
(b) 2^x
ANSWER:
(b). Exponential growth functions grow faster than power functions.
COMMENT:
You could ask about the behavior as $x \to -\infty$ as well.

5. **(a)** $10(2^x)$
(b) $72{,}000x^{12}$
ANSWER:
(a). Exponential growth functions grow faster than power functions, no matter how large the coefficient.
COMMENT:
One reason for such a question is to note that global behavior may not be determined by local behavior.

6. List the following functions in order from smallest to largest as $x \to \infty$ (that is, as x increases without bound).

(a) $f(x) = -5x$
(b) $g(x) = 10^x$
(c) $h(x) = 0.9^x$
(d) $k(x) = x^5$
(e) $l(x) = \pi^x$

ANSWER:
(a), (c), (d), (e), (b). Notice that $f(x)$ and $h(x)$ are decreasing functions, with $f(x)$ being negative. Power functions grow slower than exponential growth functions, so $k(x)$ is next. Now order the remaining exponential functions, where functions with larger bases grow faster.
COMMENT:
This question was used as an elimination question in a classroom session modeled after “Who Wants to be a Millionaire?”, replacing “Millionaire” by “Mathematician”.