Pile Splitting Problem: Introducing Strong Induction

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Summary

In many textbooks in discrete mathematics there are numerous examples for teaching the Weak Form of the Principle of Mathematical Induction, but relatively few elementary problems for applying the Strong Form. What follows is a nice example to draw on when introducing the strong form. It can be presented as a classic puzzle, it has a number of variants and it is inherently recursive.

By introducing the problem (Pile Splitting) as a puzzle, the instructor can engage the students in the process of finding a general solution. She can, then, raise the question as to how they can demonstrate that their conjecture is correct, and, thereby, motivate the need for strong induction. After an induction argument has been presented (the Worksheet includes one such proof), variants of the puzzle can be assigned for the students to work on in class or as a homework assignment.

Notes for the instructor

To give students practice in making conjectures about the solution to the puzzle, they should be asked to solve it themselves. One hands-on approach that works well is to provide each student with a sufficient number of beads or pennies for her to actively play the game enough times with different values of n so as to see a pattern emerge. Those students who correctly conjecture the general solution can assist the others. One way of reaching the conjecture is explained on the Solutions page. During the induction phase a template or "script" for a correct induction argument should be presented to the students (and they should be required to follow the script) so that they become more confident in using the technique and more convinced logically that induction does what we claim it does. The Worksheet includes a complete proof for one puzzle variant in order to give students another example.

In addition to the four variants of the puzzle described in this paper, there are a number of others which can be found in [2]. It should be noted that the author of that article presents a different approach which does not explicitly make use of mathematical induction to prove the general solutions correct.

The recursive nature of the pile splitting problem can lead to a discussion of recursive definitions, recurrence relations, techniques for solving recurrence relations and constructing recursive algorithms to compute the solutions. In the latter case, strong induction comes into play again. It addresses the question: "Does such and such a recursive algorithm, which is designed to compute something, actually do so?"

Bibliography

[1] Rosen, Kenneth. *Discrete Mathematics and Its Applications* 5th ed., McGraw Hill, 2003.

[2] Tanton, James. "A Dozen Questions About: Pile Splitting," *Math Horizons* 12 (2004) 28–31.

Worksheet on the Pile Splitting Problem

Here is a statement of a common version of the pile splitting problem.

> Given n objects in a pile, split the objects into two smaller piles. Continue to split each pile into two smaller piles until there are n piles of size one. At each splitting, compute the product of the size of the two smaller piles. Once there are n piles, sum all the products computed. The result will always be the same no matter how each of the piles is split into two smaller piles. The sum of products is a function of n. Conjecture what the sum is and prove the conjecture correct. [1]

The solution to this pile splitting problem is $(n^2 - n)/2$. Playing with a few examples can provide the necessary insight to come up with the general solution. Applying the strong form of the principle of mathematical induction can demonstrate the correctness of the conjecture, and, *equally as important, that the computation will always produce the same result no matter how the piles are split*.

For the following variant on the standard pile splitting problem, a complete proof of the general solution is provided below.

> We begin with a pile of n objects and proceed to reduce the pile to n piles of size one in the manner described above. Suppose at each splitting the sizes of the two smaller piles are labeled r and s. Now, instead of just computing the product $(r \cdot s)$ of the size of each pair of split piles, we compute the following product: $(r \cdot s) \cdot (r + s)$. And at the end of the process we add all these products. Again, the sum of products turns out to be a function of n: $(n^3 - n)/3$.

Here is the induction argument.

Let $P(n)$ be: for a pile consisting of n stones and split according to the rules above, the sum of all products of the sum and product of each pair of split piles is $(n^3 - n)/3$.

Show $(\forall n \geq 1)\ P(n)$.

Basis Step Show $P(n)$ is true when $n = 1$.

When $n = 1$, there are no splits; hence the product of sums and products is 0. For the formula, when $n = 1$,

$$\frac{n^3 - n}{3} = \frac{1^3 - 1}{3} = 0.$$

Inductive Step Suppose for any $k > 1$ that $P(1), P(2), \ldots, P(k-1)$ are true. Show this implies that $P(k)$ is true; that is, suppose any pile of j stones where $1 \leq j \leq k-1$, the sum of all products of the sum and product of each pair of split piles is $(j^3 - j)/3$. Show that this implies the sum of the products of the sums and products is $(k^3 - k)/3$.

First, divide the pile of k stones into two piles of j and $k - j$ stones. Then, the sum of all the products of sums and products equals the sum of the product of $j + (k - j)$ and $j \cdot (k - j)$ along with all the remaining sums. However, since both j and $k - j$ are between 1 and $k - 1$, the induction hypothesis applies and the sum of the products of sums and products is

$$\begin{aligned}
&= (j + (k - j))j(k - j) + \frac{j^3 - j}{3} + \frac{(k - j)^3 - (k - j)}{3} \\
&= \frac{3(jk^2 - j^2k) + j^3 - j + (k - j)^3 - (k - j)}{3} \\
&= \frac{3jk^2 - 3j^2k + j^3 - j + k^3 - 3jk^2 + 3j^2k - j^3 - k + j}{3} \\
&= \frac{k^3 - k}{3}.
\end{aligned}$$

Thus, since both the **Basis Step** and the **Inductive Step** have been shown to be true, $(\forall n \geq 1)\ P(n)$.

Additional Questions

Consider the following two variations on splitting a pile of n objects. In each case try a few examples, conjecture what the solution should be (again, it turns out to be a function of n) and then use the strong form of the principle of mathematical induction to prove your conjecture. As in the problem statement above, suppose at each splitting the sizes of the two smaller piles are r and s.

1. Split the pile of n objects according to the rules above. At each splitting compute the sum of the reciprocals of the two smaller piles: $(\frac{1}{r} + \frac{1}{s})$. Once there are n piles, multiply all of the sums computed. Again, the result will always be the same no matter how each of the piles is split into two smaller piles.

2. Split the pile of n objects according to the rules above. At each splitting compute the following combinatorial number: $\binom{r+s}{r}$. Once there are n piles, multiply all the combinatorial numbers computed.

Solutions

- For the problem in the worksheet, where at each stage the product $(r \cdot s) \cdot (r + s)$ is computed, it turns out that, for piles of size $1, 2, \ldots, 8$, the sum of all the products is 0, 2, 8, 20, 40, 70, 112 and 168, respectively. Computing the difference of consecutive pairs of integers in the sequence leads to the seven integers 2, 6, 12, 20, 30, 42 and 56 and to the recurrence relation $a_n = a_{n-1} + n(n-1)$ for $n \geq 2$ with initial condition $a_1 = 0$, the solution to which is $(n^3 - n)/3$.

  At this point it could be said that a solution to the pile splitting problem has been identified and proved correct without using the strong form of induction. That is true. However, it turns out that in most discrete mathematics courses induction is introduced before techniques for solving such linear recurrence relations. If this is the case, after the students have worked on the problem for a while, the instructor might provide some hints or even make a claim as to the general solution and, then, ask the students to use strong induction to prove that claim. (It would be instructive to see if some students can come up with the correct conjecture without reference to recurrence relations and explain their reasoning to the class.)

- For the two puzzles in the Additional Questions Section the solutions are n and $n!$, respectively.