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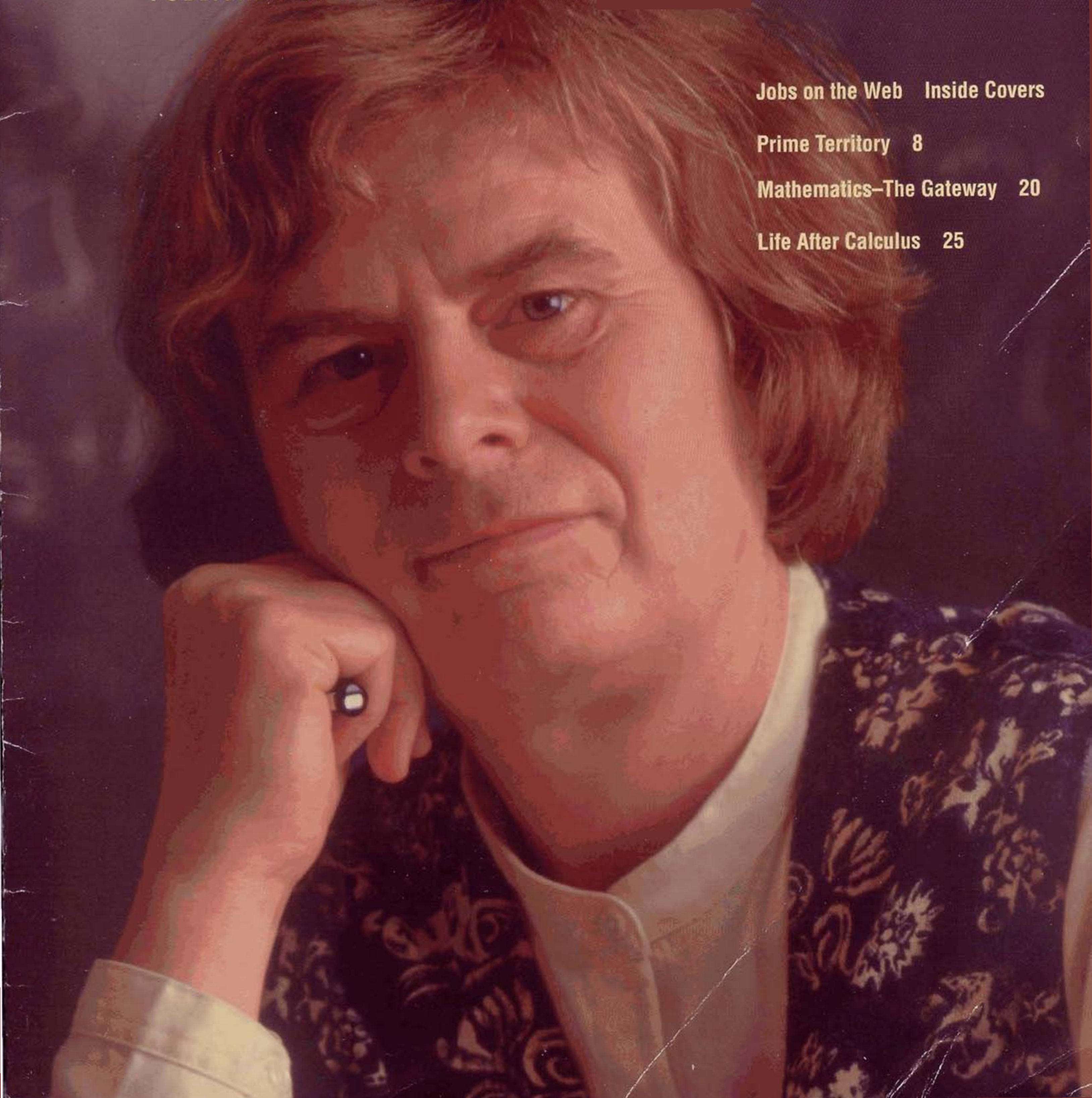

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EDITORIAL

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DAN KALMAN

A Perfectly Odd Encounter in a Reno Cafe

My father and I were sitting in a cafe in Reno. I was giving him some examples from number theory, including a problem that has been unsolved since the days of ancient Greece. Unintimidated, my father came up with a solution in about a minute flat. The reasoning behind his solution, and my skeptical reaction, reveal something about a central concern of mathematics: the nature of proof.

We were killing time while others in our party were playing the slot machines. Ted, that's my father, had been talking about the mathematics in a book he was reading, Michener's *The Source.* In fact, it wasn't really mathematics at all. It was numerology, the mystical interpretation of numerical relationships for purposes of divination. It occurred to me that number theory is the closest thing in real mathematics to the numerology Ted had been talking about, and I tried to describe the subject.

As an example, I told him about perfect numbers. The number 6 is perfect, because if you add up its proper divisors, 1, 2, and 3, the total is 6. Another example is 28: the proper divisors are 1,2,4, 7, and 14, and these sum to 28. Are there any others? Can you find some?

It has been known for over 200 years how to find all the even perfect numbers. There is a formula: $2^{n-1}(2^n-1)$. For $n=2$ this gives $2^1(2^2-1)=2\cdot 3=6$. For $n=3$ we get $2^2(2^3-1)=4\cdot 7=28$. The next possibility, $n=4$, yields 120, and that isn't a perfect number because 60, 40, and 30 are all divisors. The trouble is that for $n=4$, (2^n-1) isn't a prime number. Euclid showed that when (2^n-1) is prime, the formula $2^{n-1}(2^n-1)$ always produces a perfect number. Thus, for $n=5$, $2^n-1=31$ is prime, so we can be sure that $2^{n-1}(2^n-1)=496$ is perfect. Some two thousand years later, Euler proved that Euclid's formula actually generates all the even perfect numbers. So today we know the complete story on even perfect numbers.

> *Number theory…is bursting with curious relationships that aren't particularly good for anything, but which have fascinated… mathematicians for centuries.*

So what? Who cares? What possible use could there be in knowing about perfect numbers? Well, number theory is like that. It is bursting with curious relationships that aren't particularly good for anything, but which have fascinated amateur and professional mathematicians for centuries. The number theorist and the numerologist share this fascination with numbers, but the number theorist doesn't try to draw mystical conclusions from the number patterns. The object is simply to understand the mysteries and to back up each insight with proof.

Of course, it isn't always easy to find proof. That is why number theory abounds with conjectures: that is, statements fitting all the known data, and seeming to be valid general laws, but for which no proof has been found. Number theorists do not despair of ever finding proofs for these conjectures. Why, Fermat's last theorem was recently proved after standing as a conjecture for 350 years. Fermat wrote in the 1640's that $x^n+y^n=z^n$ could never hold for positive integers x, y, z, and n, with $n>2$. That is, when working with positive integers, the sum of two cubes is never a cube, the sum of two fourth powers is never a fourth power, and so on for all powers greater than 2. From Fermat's day until our own, no proof could be found for his statement. But in 1993, Andrew Wiles announced that he had discovered such a proof, and today it is generally accepted that Fermat's theorem has been established. So, number theorists continue to hold out for proofs. No matter how overwhelming the evidence, no matter how clear the insight, true understanding is not conceded until there is proof.

That is exactly the situation with odd perfect numbers. Since Euclid's time, no one has ever been able to find an odd perfect number, even though the numbers checked by computer reach

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Illustration by Marty Tatum of Ice House Graphics

into the millions and beyond. It seems like an inescapable conclusion that odd numbers simply cannot be perfect and perfect numbers simply cannot be odd. But no one has been able to prove it.

So there I was, explaining to Ted about number theory, how it is like numerology, and how it is unlike. To illustrate the ideas of conjecture and proof, I told him about perfect numbers commenting that one of the oldest open questions in number theory is whether there are any odd perfect numbers. I told him, as I have told you, that the even case is completely solved. I described the state of affairs for the odd case: no one can find an odd perfect number, yet no one can prove that none exist.

Then, to my astonishment, Ted announced that it was completely obvious that an odd perfect number is an impossibility. He explained his reasoning this way: An even number is divisible by 2, and when you divide it by 2, you get one of its divisors. In fact, you get its largest possible proper divisor, half of the original number. For an even perfect number, adding up the remaining divisors produces the other half of the original number. But if the original number (n) is odd, the smallest factor (other than 1) is at least 3, so the largest proper factor is at most $n/3$. In that case, in order for n to be perfect, the other divisors—all of which are even less than $n/3$—have to add up to $2/3$ of n, and that is impossible.

Is that a proof? Was the famous problem of odd perfect numbers solved in a Reno cafe? I was instantly skeptical. Surely this argument could have occurred to Gauss, or Euler, or even *me*. But even more compelling, just from its inherent structure, I instantly realized that Ted's argument was not a proof. Can you see why?

One of the foremost skills of the trained mathematician is to recognize what is a proof, and what is not. Yet it is not always easy to clearly explain what constitutes a proof. My father's argument is logical, it is insightful, it seems to explain things. And yet there is a huge hole, a gap in the reasoning. *Why* is it impossible for there to be enough small factors to total $2/3$ of the original number? Ted could give no further explanation.

On the surface, the nature of proof seems clear cut. There must be a logical reason for each conclusion. If any of the conclusions is questioned, the prover must be able to provide reasoning that justifies it. This additional reasoning, in turn, is open to challenge, and must likewise be defended. And so on, and so forth, the prover must be prepared to provide a justification for each conclusion that is questioned. But this process cannot be taken infinitely far. At some point, won't the prover be reduced to the same position as my father? At some point, a step will be reached that is so self evident that no further explanation can be advanced. The prover can only insist that the skeptic must surely agree with the conclusion, just as my father insisted that no further argument was needed for his proof. His conclusion was transparently self evident! It was obvious! This is where deciding what is a proof gets tricky. It comes down to recognizing what is obvious, and what isn't.

Well, how *does* one recognize the obvious? It reminds me of what the Supreme Court justice said about pornography. I may not know how to define it, but I know it when I see it. Ted's final assertion was definitely not obvious. His inability to explain further invalidated the proposed proof. By training and long habit of thinking, my eye is instinctively drawn to potential gaps or holes in arguments. I constantly probe the fabric of a proof. Is there a weakness here? Can I argue a point further there? It is my years of practice that qualify me to judge what is obvious. All mathematics students need to work at this same kind of skepticism. Be wary of the obvious, distrust it, be as obtuse as you possibly can. If it is overly obvious, one ought to be able to explain why. Dig deeper, push harder, however brightly lit the corner, shine an even brighter light there, until the shadows are driven out utterly. Make this your standard practice. Only then will you be qualifed to say what is obvious.

Of course, there is little satisfaction in simply denying what someone else claims is obvious—far better to demonstrate that the desired conclusion need not follow. In the case of Ted's argument, this can be accomplished by considering the example of 945. That is an odd number whose proper divisors sum to 975. Although the largest divisor is 945/3 = 315, and all of the other divisors are even smaller, there are enough of these small divisors to add up to more than 2/3 of the original number. This is just what my father's argument claimed was impossible. This example highlights the flaw in his reasoning. And since it is possible for the proper divisors of an odd number to have a sum that exceeds the number, it might also be possible for an odd number to be perfect.

At this point, you are probably asking where the 945 came from. How did I find this example? I went looking for it. I noticed that Ted's argument, if valid, would not only rule out the possibility of odd perfect numbers, but would also rule out the existence of an odd number which is *exceeded* by the sum of the proper divisors. Also, I knew a useful fact from number theory: Given the prime factorization $m = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, the sum of all the divisors, including m itself is equal to

$$(1 + p_1 + p_1^2 + \cdots + p_1^{e_1}) \times$$
$$(1 + p_2 + p_2^2 + \cdots + p_2^{e_2}) \times \cdots \times$$
$$(1 + p_n + p_n^2 + \cdots + p_n^{e_n})$$

Illustration by Marty Tatum of Ice House Graphics

It is easy to see that this is true. Simply observe that when the product is multiplied out, the terms are all the divisors of m. Things can be simplified a little by applying the formula

$$(1 + p_i + p_i^2 + \cdots + p_i^{e_i}) = \frac{p_i^{e_i+1} - 1}{p_i - 1}$$

This gives the sum of the divisors as

$$\frac{p_1^{e_1+1} - 1}{p_1 - 1} \frac{p_2^{e_2+1} - 1}{p_2 - 1} \cdots \frac{p_n^{e_n+1} - 1}{p_n - 1}$$

Using this last expression, and a hand held calculator, it is nearly effortless to compute the sum of the divisors for any number. For example, if

$$m = 3^4 \cdot 5^2 \cdot 13$$

then the sum of the divisors is $(242/2)(124/4)(168/12) = 121 \cdot 31 \cdot 14$. That's a pretty nifty way to total up the divisors of 26325!

Remember the goal is to find an odd number m which is smaller than the sum of its *proper* divisors, that is, all the divisors other than m itself. Put another way, we need the sum of all the divisors of m to exceed $2m$. Well, experiment a bit. Remember to use odd numbers, and consider some m's with two or three different prime factors, some with exponents greater than 1. It does not take long to stumble on the example

$$3^3 \cdot 5 \cdot 7 = 945.$$

And what happened to my father? Did he change his mind about odd perfect numbers? I am sorry to report that he did not. By the time I found my example, a day or two had gone by and he had moved on to other things. I don't think he recalled what the main thread of his argument had been, or indeed, what the entire dispute was about. Reno, after all, has many other diversions, and one's relatives can only be expected to sit still for so much mathematics. But I hope this article has contributed to your own understanding of proofs. And the next time something appears obvious, think of my father and the cafe in Reno where, from his point of view, the non-existence of odd perfect numbers was proved. ■

I am grateful to William Dunham of Muhlenberg College for suggesting many improvements to this paper.

ELLEN GETHNER

In Prime Territory

A Math Question Posed by Michael Jordan

Did you know that Chicago Bulls player Michael Jordan used to be a math major at the University of North Carolina before switching over to geography? We're probably better off not discussing the moral of *that* story. In any case, during the 1994–95 basketball season, MJ renewed his interest in mathematics when he changed his new team number (45) back to his old and formerly retired team number (23). He posed the following question: 'What's in a number, anyway?' MJ was quite likely wondering about *prime numbers*.

What is a prime number anyway? Let's review. First, *a whole number*, or *counting number*, is a number such as 1,2,3,4,5,6,7,... and so on, forever. A *prime number* (or just *prime* for short) is a whole number that is divisible **only** by 1 and itself. For example 2,3,5,7,11,13,17,19 and 23 are primes, but $4 = 2 \times 2$, $6 = 3 \times 2$, and $45 = 5 \times 3 \times 3$ are not primes. In fact, numbers that are **not** primes are called *composites*.

For later use, let's expand the set of whole numbers by including 0 and the negatives of the whole numbers, like, 0, –1, –2, –3, –4, –5, –6, –7,..., and so on. This new expanded set is known as the *integers*.

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Photograph courtesy of the Chicago Bulls and Bill Smith

#23 Michael Jordan, a prime player.

What Euclid Knew About Primes

Before moving on to further technicalities, it might be interesting to take a historical perspective on the prime numbers. In particular, Euclid was a great mathematician who lived in Alexandria during the 3rd Century, B.C. and who wrote a thirteen-volume work called *Elements*. The purpose of *Elements* was to compile all of the known geometry of the time into one comprehensive work.

In volume IX Euclid asked if there were infinitely many primes. To the uninitiated, the answer seems to be a straightforward and **resounding 'yes,'** simply because there are infinitely many whole numbers. But, mathematicians need a more convincing (and for that matter correct!) argument, as we shall see.

Euclid did, in fact, prove that there are infinitely many primes and here's the idea behind his argument. First assume there are only a finite number of primes in the world. For example, suppose 2 and 3 are the only primes, and the rest of the whole numbers are composite (humor me). Now think about the number

$$2 \times 3 + 1 = 7$$

Well, 7 isn't supposed to be prime (remember, 2 and 3 are the *only* primes for the time being), which means that 7 must be divisible by smaller primes. Well, our only choices are 2 and 3. But dividing 7 by either 2 or 3 leaves a remainder of 1, which means that 7 is neither divisible by 2 nor by 3. What does this mean? Our assumption was incorrect, and therefore, 2 and 3 cannot be the only primes in the world.

The above paragraph was only a warm-up to Euclid's real argument. Here is Euclid's more general argument to prove that there are infinitely many primes. For a contradiction, assume that there are only finitely many primes. Here they are: 2, 3, 5, 7,..., p, where p is the largest prime (in the warm-up argument, $p = 3$). Now multiply all of the existing primes together and add 1 to the result:

$$(2 \times 3 \times 5 \times 7 \times \ldots \times p) + 1.$$

The above gigantic number cannot be a prime (remember, p is the largest prime) and therefore must be divisible by one or some of our primes, namely 2,3,5,7,...,p. But dividing the gigantic number by any of these primes leaves a remainder of 1. This is the contradiction that Euclid was seeking, namely that the gigantic number is neither prime itself nor is it divisible by any of the finitely many primes in the world. So, our assumption was wrong. In particular, there must be infinitely many primes.

Take A Hike!

Now that we've got all of these primes, what should we do with them? Try the following game. Stand on the number 0 and start walking. The only rule to this game is, as you take each step, you are **only** allowed to step on the primes. A curious question: If you keep walking forever, will you, every once in a while, have to take longer and longer steps? Or, might you be able to keep walking and taking only 'small' steps. This is how the experts ask the above question: Can you walk to infinity on the primes in steps of bounded length? The answer is 'no,' and here's why.

In order to show that a prime-hiker will eventually have to take longer and longer steps, it suffices to show that we can find an arbitrary number of composites in a row. For example, if we were to find 4 composites in a row, a prime-

Euclid

hiker would be forced to take a step of length 5. More abstractly, if we were to find n composites in a row, then the prime-hiker would be forced to take a step of length $n + 1$. But let's start out slowly and aim for finding 4 composites in a row. *Voilá*–here they are:

$$5 \times 4 \times 3 \times 2 + 2 = 122,$$

$$5 \times 4 \times 3 \times 2 + 3 = 123,$$

$$5 \times 4 \times 3 \times 2 + 4 = 124,$$

$$5 \times 4 \times 3 \times 2 + 5 = 125.$$

The first number, 122, is divisible by 2, the second number by 3, the third by 4, and the fourth by 5, and hence the prime-hiker will be forced to take a step of length (at least) 5.

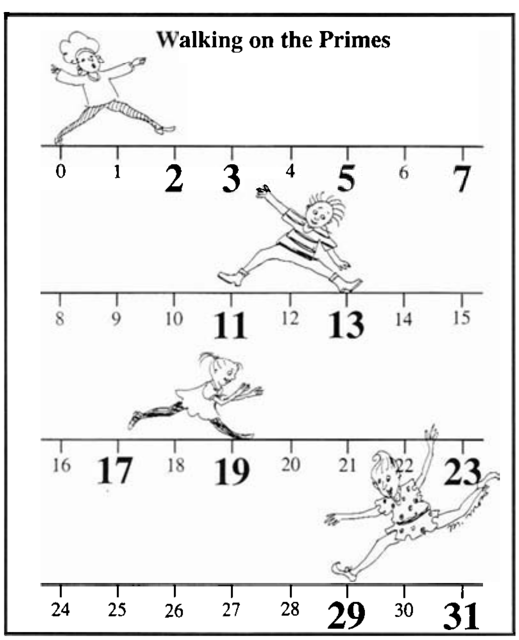
Now let's be more ambitious. Let's find 99 composites in a row to force the poor hiker to take a step of length (at least) 100. But first we need help with the notation because we'll have to multiply lots of numbers together. In particular, the number

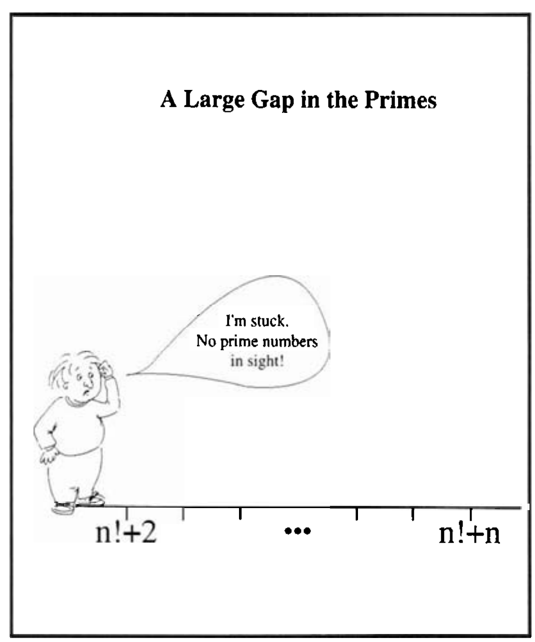
$$100 \times 99 \times 98 \times \ldots \times 3 \times 2 \times 1$$

is more easily written as 100! and is pronounced 'one hundred factorial.' If you are unfamiliar with this notation, you might at first think that 100! means that 100 is a very exciting number, but remember, one hundred factorial means 'multiply all the whole numbers less than or equal to 100 together.' Now we're ready. Here are 99 composites in a row:

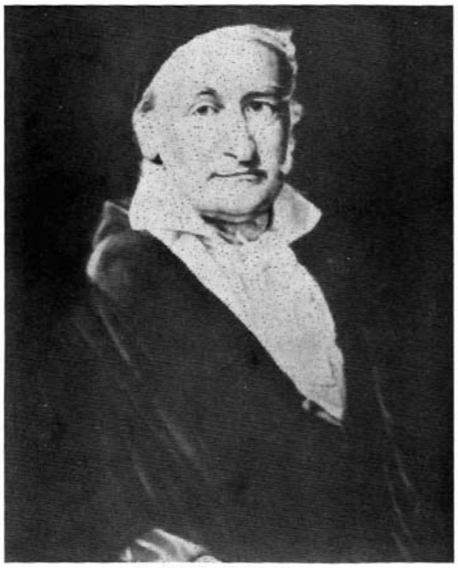
$$100! + 2,$$
$$100! + 3,$$
$$100! + 4,$$
$$\ldots$$
$$100! + 99,$$
$$100! + 100.$$

Got it? The first number, 100! + 2 is divisible by 2, the second number





Illustrations by Marissa Moss



Carl Friedrich Gauss

100! + 3 is divisible by 3, and so on up to the last number 100! + 100 which is divisible by 100. In all there are 99 composites in a row, which forces the prime-hiker to take a step of length (at least) 100. By now perhaps you see the way the general construction works. Specifically, for any n, you can force the prime-hiker to take a step of length (at least) n by producing the following $n-1$ composites in a row ($n!$ means multiply all whole numbers less than or equal to n together):

$$\begin{array}{l} n!+2, \\ n!+3, \\ n!+4 \\ \quad\dots \\ n!+(n-1) \\ n!+n. \end{array}$$

All in all we've learned that one cannot walk to infinity on primes in steps of bounded length.

C.F. Gauss Visits the Planet Vulcan

Carl Friedrich Gauss was a famous 18th century German mathematician whose interests were very diverse.

In fact, mathematics on the planet Earth was not enough to keep Gauss interested (ahem, the author begs pardon), and so he visited the planet Vulcan to learn how the inhabitants count. As in any respectable math paper, we need a good example. How about Mr. Spock of Star Trek fame?

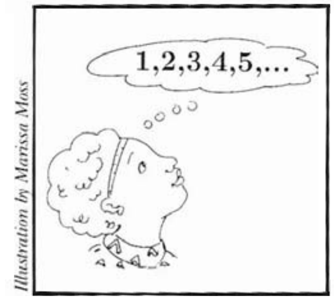
A bit of background is in order here. Spock was born on the planet Vulcan in the year 2230. His father, Sarek, was a Vulcan diplomat, and his mother, Amanda Grayson was a human scientist. So, just how *does* Spock count? Think about it. When you were a kid and were first learning how to count, what did you do? You held up your hand and pointed to your fingers. Eventually, you could very smugly count in your head.

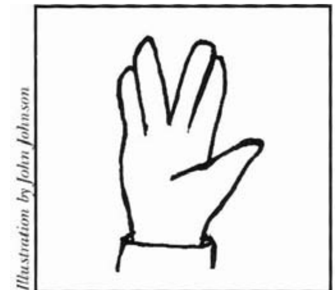
What does Spock do when *he* counts? When Spock was a kid, he held his hand up and, well, you know what *that* looked like.

Apparently, Spock counts in *pairs* of numbers, in particular, in pairs of integers.

About the Gaussian Integers (or Spock's Numbers)

Spock's numbers, known as *Gaussian integers* to mathematicians, are pairs of integers like (7, 12), (–2, 6), (13, 4), (–206, 10027), and so on, forever. The picture of all Gaussian integers is too large to fit on this page (there are infinitely many Gaussian integers!), but a small portion of the picture is shown in figure 2 and is a grid made up of many small squares.





Counting in pairs

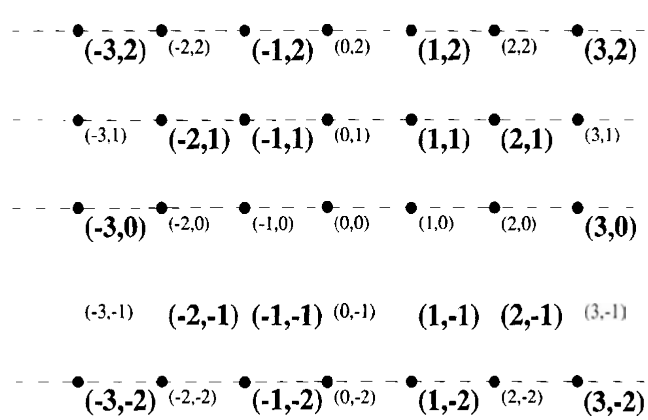
Another way of describing the Gaussian integers (and this is what Gauss himself did) is to write, for example, (2,3) as $2+3i$ and (1, 4) as $1+4i$, where $i^2=-1$. Then multiplying Gaussian integers is just as you think it should be. That is (2,3) × (1,4) is

$$(2+3i)\times(1+4i)=2+8i+3i+12i^2=-10+11i$$

(or (–10,11)). The general method for multiplying two Gaussian integers, say (a,b) and (c,d), is the same as above:

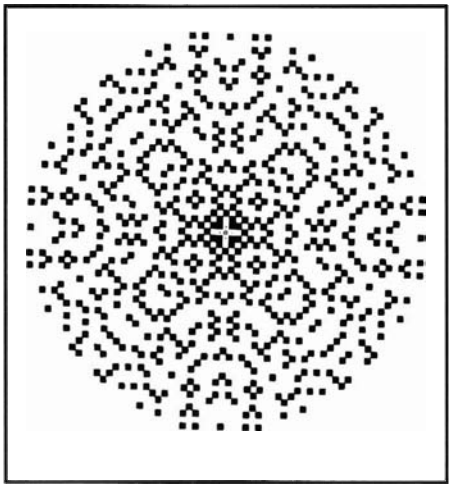
$$\begin{aligned}(a+bi)\times(c+di) &= ac+adi+bci+bdi^2 \\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

(or $(ac-bd,\ ad+bc)$). Once we have this method, it makes sense to talk about Gaussian *primes.* That is, a Gaussian integer is a Gaussian prime exactly when it can't be written as a product of 'smaller' Gaussian integers, where 'smaller' means closer to the origin, (0,0). This amounts to the condition that (a,b) is smaller than (c,d) if $a^2+b^2<c^2+d^2$. The

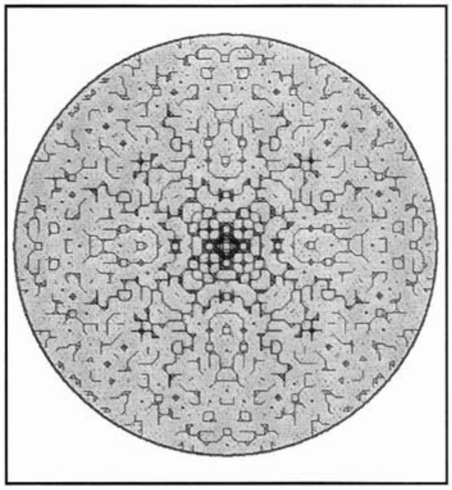


Spock's Numbers (also known as Gaussian Integers)

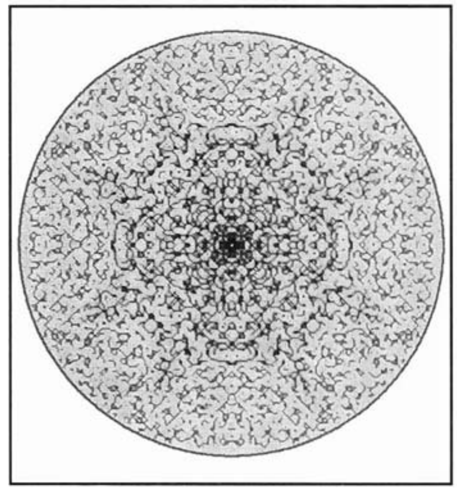
Figure 2



Some Gaussian Primes



Gaussian 2-moat

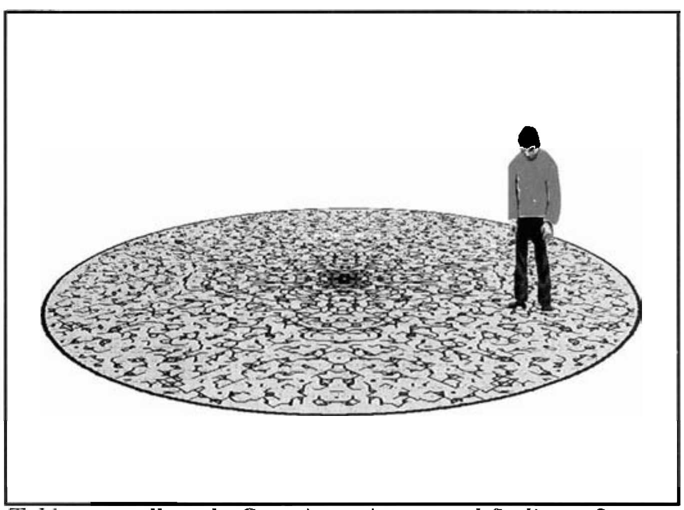


Gaussian 3-moat

Gaussian integers that appear in boldface in figure 2 are actually Gaussian primes.

A Quick Trick

There is a nice trick for determining exactly when a Gaussian integer is prime. Suppose someone hands you the Gaussian integer (5,2) and wants to know if this number is a Gaussian prime. What to do? You first compute $5^2 + 2^2 = 25 + 4 = 29$, and check to see if the result is a (regular) prime. In this case, 29 is prime, so you're in luck: (5,2) is a Gaussian prime. This trick works on all Gaussian integers (a,b) as long as neither one of a nor b is zero. That is, given a Gaussian integer (a,b), compute $a^2 + b^2$. The result is a (regular) prime precisely when (a,b) is a Gaussian prime.



Taking a stroll on the Gaussian primes...and finding a 3-moat.

To finish off this trick, we need to know what to do for the Gaussian integers $(a, 0)$ or $(0,b)$ (remember, the trick only works, so far, for (a,b) when neither a nor b is 0.) The answer is that $(a,0)$ (respectively $(0,b)$) is a Gaussian prime precisely when $|a|$ (respectively $|b|$) is a regular prime which is exactly 3 greater than a number divisible by 4. For example, (11,0), (–11, 0), (0,11), and (0, –11) are all Gaussian primes, whereas (–5, 0), (5,0), (0,5), and (0,–5) are Gaussian composites. (For the skeptics in the audience, what happens when you compute (2,–1) × (2,1)?) As usual, a picture is worth 10,000 words, and the graphic above and on the left shows all Gaussian primes within a distance $\sqrt{1000}$ from the origin, (0,0)

Take Another Hike!

Now that we've got all of these Gaussian primes, what should we do with them? Take another hike, of course. We'll try the prime-hiking game on the Gaussian primes this time. Here goes: Stand on the number (0,0) and start walking. Remember, you can **only** step on Gaussian primes. The question is, if you keep walking forever, will you, every once in a while, have to take longer and longer steps? Or might you be able to keep walking and taking only 'small' steps? Now that you're an expert, you would ask: Can you walk to infinity on the Gaussian primes in steps of bounded length? The answer? Nobody knows...(dramatic pause)...yet.

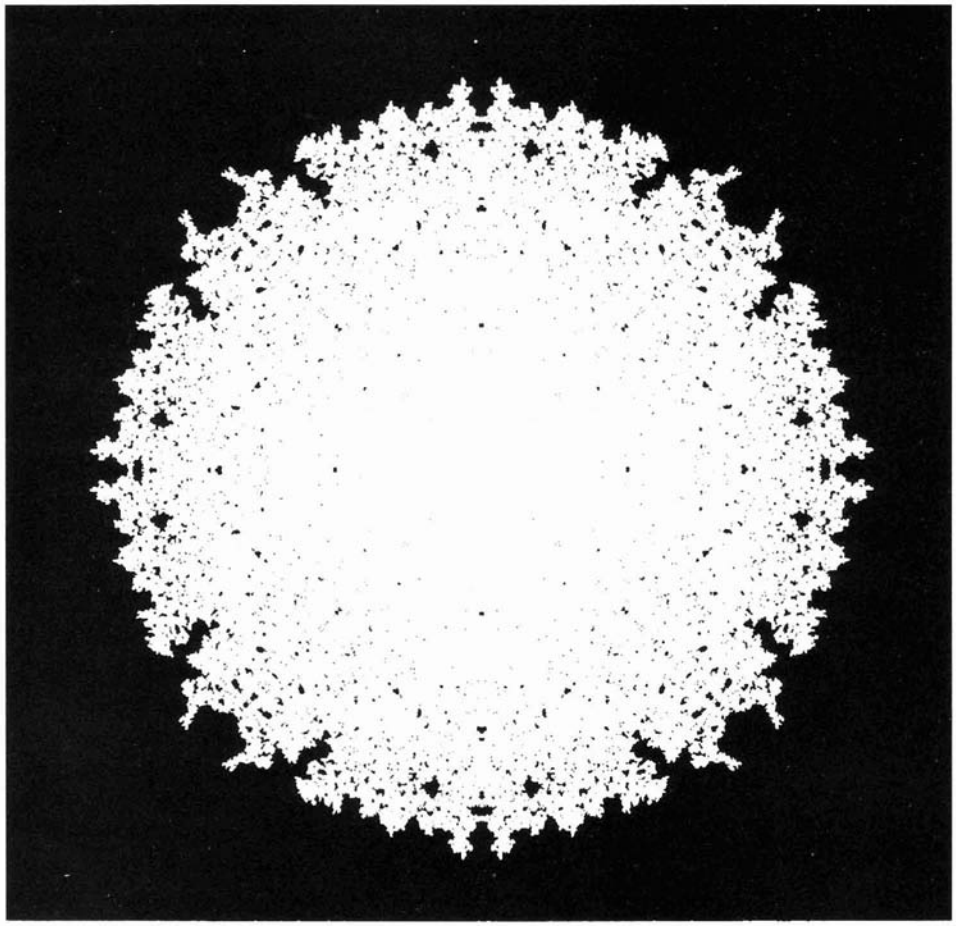
A GIANT Homework Problem

The Gaussian prime walking problem was posed in 1962 by the mathematician Basil Gordon from UCLA. It remains unsolved to this day. The truth is, the answer isn't in the back of any textbook, and isn't outlined in any paper. It just isn't known! (The author happily admits to having spent more than 1.5 years looking for the elusive solution, and expects to spend a great deal *more* time doing so.) This problem is a research problem, and is just one of a plethora of such unsolved or open problems.

"Moativation"

The usual image that comes to mind when one thinks of a moat is a body of water surrounding a castle to keep out intruders. In our case the center of the castle is at the origin and the moat is a squiggly band (of nonconstant width) completely made up of Gaussian composites. Here is an idea for solving the Gaussian prime walking problem. If we can keep finding 'fatter and fatter' Gaussian moats

Brian Wick, using Mathematica, produced a condensed picture of a $\sqrt{18}$–moat. His method was to look at 25 x 25 blocks of Gaussian integers and reduce each down to a single point.



Gaussian $\sqrt{18}$*-moat*

surrounding the origin, then the prime-hiker will have to keep taking longer and longer steps. But, as usual, pictures are worth multitudes of words thanks to Stan Wagon (Macalester College) and Brian Wick (University of Alaska, Anchorage).

Stan Wagon, using *Mathematica*, programmed the Gaussian 2-moat shown on page 11, by observing the following rules: connect two Gaussian primes with a black line if a) they are at most distance 2 apart, and b) either prime can be reached from the origin in steps of length at most 2. Connect two Gaussian primes with a green line if a) they are at most distance 2 apart, and b) neither prime can be reached from the origin in steps of length at most 2.

So, for example, the unfortunate prime-walker who thought she or he could walk to infinity in steps of length at most 2 would start following the black paths and then get stuck at the moat of Gaussian composites that lies between the black lines and the green lines.

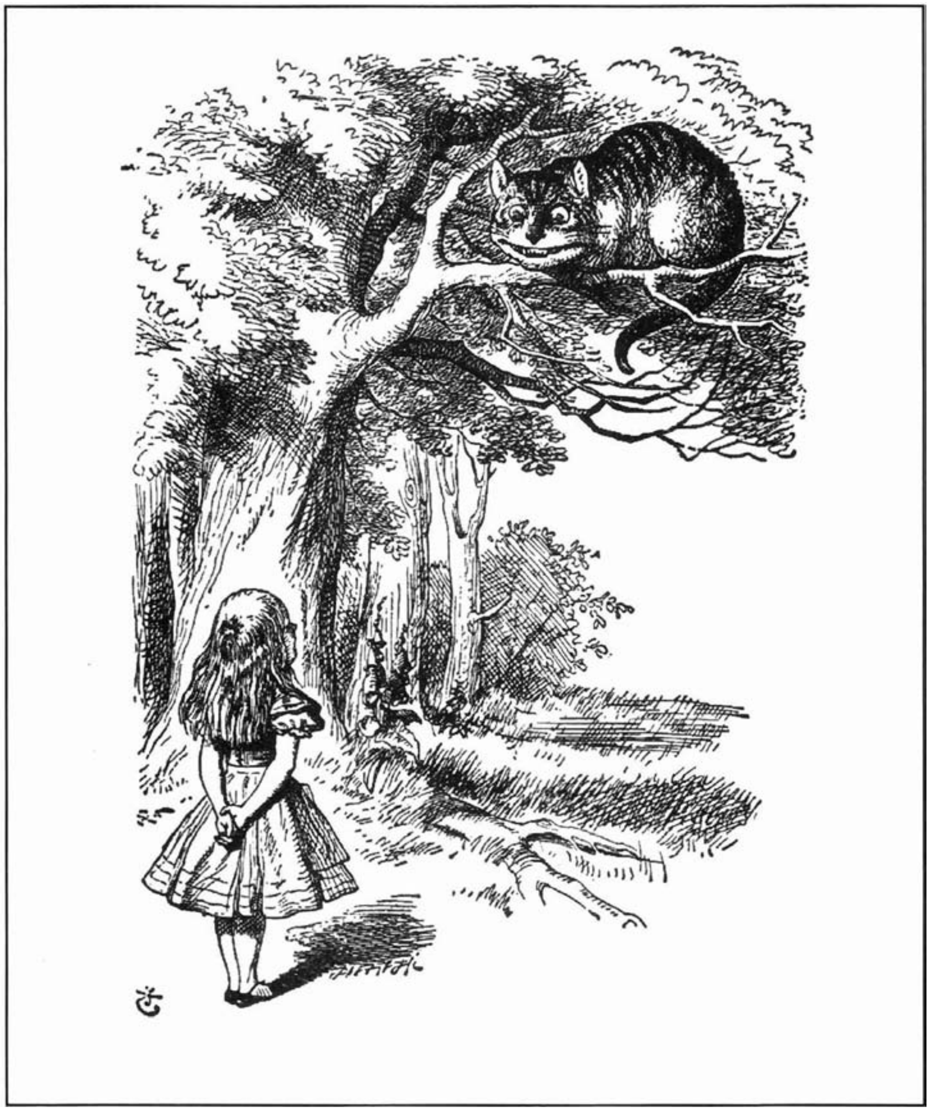
Stan Wagon used the same technique to generate a Gaussian 3-moat.

Perhaps Spock, a very clever fellow, thought he could walk to infinity in steps of length at most 3. He started walking along the network of black paths, and discovered that all of these paths were dead-ends. In other words, he ran into a Gaussian 3-moat. To get from the region with black lines to the region with green lines, Spock would have to take a step of length strictly greater than 3.

The existence and location of both the 2-moat and 3-moat were known to the two mathematicians J.H. Jordan (no relation to Michael) and J.R. Rabung in 1970, and in fact, they were able even to find a $\sqrt{10}$-moat. Unfortunately, they ran out of money, and were unable to continue their search.

Brian Wick, using *Mathematica*, produced a condensed picture of a $\sqrt{18}$-moat. His method was to look at 25 × 25 blocks of Gaussian integers and reduce each down to a single point. If a block contains one (or more) Gaussian primes which is (are) reachable from the origin in steps of length at most $\sqrt{18}$ then the corresponding point is colored white. Otherwise, it is colored black. The change from mostly white to all black dramatically outlines the $\sqrt{18}$-moat. The radius of a circle containing the uncondensed moat is approximately 10,000.

Finally, using *Mathematica*, E. Gethner, S. Wagon, and B. Wick have a



"Would you tell me, please, which way I ought to walk from here?"

"That depends a good deal on where you want to get to," said the Cat.

"I don't much care where," said Alice.

"Then it doesn't matter which way you walk," said the Cat.

computational proof that a moat of width $\sqrt{26}$ must exist.

Final Stretch

Lewis Carroll, who is best known for having written *Alice in Wonderland*, was also a mathematician. Perhaps he was thinking about walking on prime numbers when he wrote:

"Would you tell me, please, which way I ought to walk from here?"

"That depends a good deal on where you want to get to," said the Cat.

"I don't much care where," said Alice.

"Then it doesn't matter which way you walk," said the Cat.

"–so long as I get *somewhere*," Alice added as an explanation.

"Oh, you're sure to do that," said the Cat, "if you only walk long enough!"

Now you know about some of the interesting progress that has been made on the Gaussian prime walking problem, but, really, the journey has just begun. ■

References

1. L. Carroll, *Alice in Wonderland*, Grosset and Dunlap, 1988.
2. Euclides, *The Elements of Euclid*, E.P. Dutton, New York, 1933.
3. E. Gethner, S. Wagon, and B. Wick, *"A stroll through the Gaussian primes,"* forthcoming.
4. J.H. Jordan and J.R. Rabung, *"A conjecture of Paul Erdös concerning Gaussian primes,"* Math. Comp., 24 (1970), 221–223.

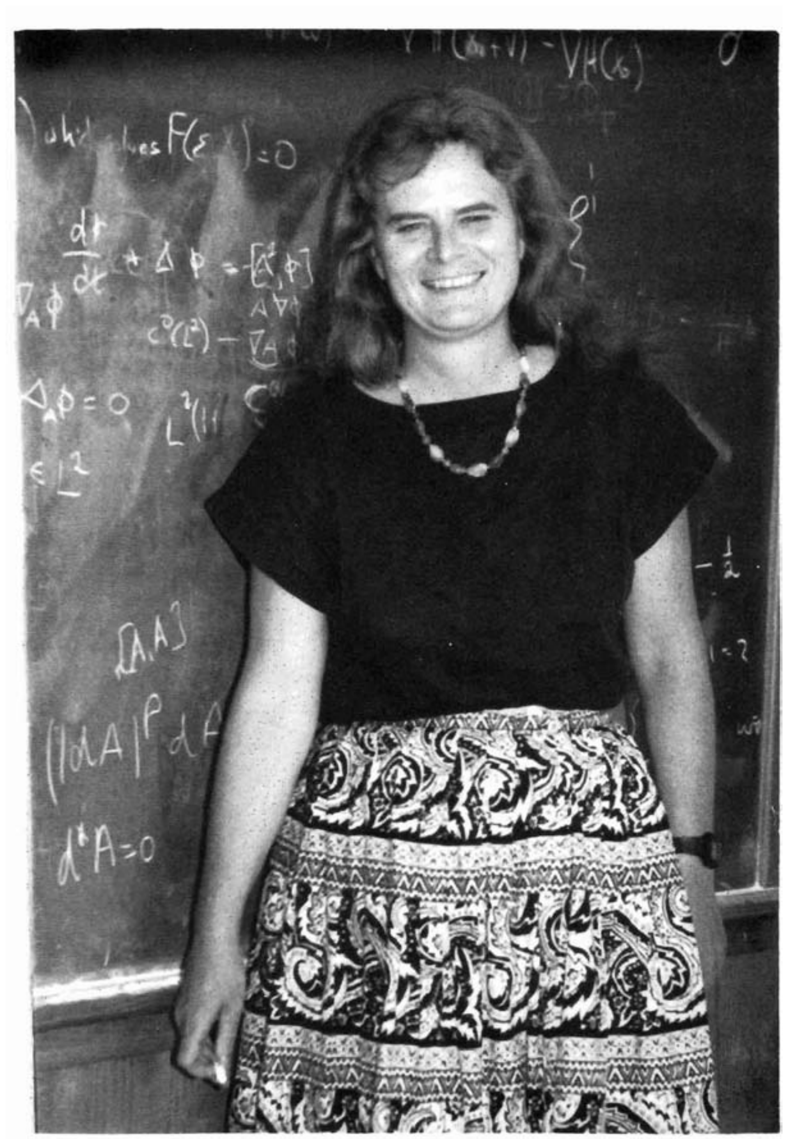
Acknowledgments

The author wishes to thank Bill Dunham, Bob Osserman, and Bill Thurston for their advice and inspiration, Marissa Moss for her illustrations, Stan Wagon for his technical help with the Gaussian 2- and 3-moats (photographic-quality images of these moats are available from Stan Wagon (wagon@macalst.edu), and Brian Wick for his technical help with the Gaussian $\sqrt{18}$–moat. The author is especailly grateful to Nancy Shaw for her computer wizardry and vast patience in the preparation of all of the graphics.

Coming to Grips With Success

A Profile of Karen Uhlenbeck

Karen Uhlenbeck is an avid lover of nature, a mathematician, and a member of the American Academy of Arts and Sciences and the National Academy of Sciences. Her interest in math arose, in part, from her preference to work alone, her natural bent for abstraction, her love of ideas, and her lack of success in undergraduate physics. Although she faced blatant sexism early in her career, she never took it personally, realizing that prejudice treats an individual as a member of a class or group instead of as a person.



In spite of the blackboard full of symbols, Karen Uhlenbeck says, "I'm very visually oriented."

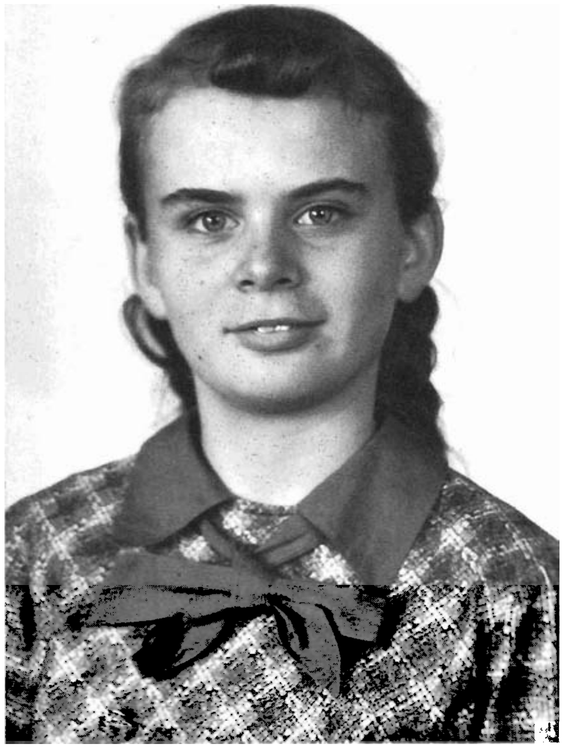
My first love is the outdoors—I enjoy mountain climbing, backpacking, hiking, canoeing, swimming, and bicycling. Many of these interests I inherited from my parents who, at age 83, are still hiking and backpacking. I am at home in nature and, when I can't be out in the wilderness, I can often be found in my garden at my home in Austin. That's the real me. My day-to-day life is something very different.

I am a mathematician. Mathematicians do exotic research so it's hard to describe exactly what I do in lay terms. I work on partial differential equations which were originally derived from the need to describe things like electromagnetism, but have undergone a century of change in which they are used in a much more technical fashion to look at even the shapes of space. Mathematicians look at imaginary spaces constructed by scientists examining other problems. I started out my mathematics career by working on Palais' modern formulation of a very useful classical theory, the calculus of variations. I de-

KAREN UHLENBECK is a professor of mathematics at the University of Texas–Austin. She is a member of the American Academy of Arts and Sciences and the National Academy of Sciences.

cided Einstein's general relativity was too hard, but managed to learn a lot about geometry of space time. I did some very technical work in partial differential equations, made an unsuccessful pass at shock waves, worked in scale invariant variational problems, made a poor stab at three manifold topology, learned gauge field theory and then some about applications to four manifolds, and have recently been working in equations with algebraic infinite symmetries. I find that I am bored with anything I understand. My excuse is that I am too poor an expositor to want to spend time on formal matters.

As a young academic I worked by myself a lot. In fact, that was one of the attractions of mathematics. I am the eldest of four children and I consider dealing with my siblings the hardest thing I've ever done in my life. That had a great impact on my choosing a career—I wanted a career where I didn't have to work with other people. I've always been competitive, but I find it difficult to cope with the attitudes of people who lose. It is still attractive to work in an area where I compete only with myself and don't have to deal with the negative aspects of competition. As my career advanced, however, I found I had a lot to learn from other people of all sorts. I have found it very rewarding to deal with younger mathematicians, and I now truly enjoy collaborative projects.



As a fifth grader, Karen often read all night long and at school under the desk!

Three-year old Karen with her parents and brother John.

I can't say that I was really interested in mathematics as a child or adolescent, mostly because one doesn't really understand what mathematics is until at least halfway through college. As a child I read a lot. I read everything, including all the books in our house three times over. I'd go to the library and then stay up all night reading. I used to read under the desk in school. My whole family were and still are avid readers: we lived in the country so there wasn't a whole lot else to do. I was particularly interested in reading about science. I was about twelve years old when my father began bringing home Fred Hoyle's books on astrophysics. I found them very inspiring. I also remember a little paperback book called *One, Two, Three, Infinity* by George Gamow, and I remember the excitement of understanding this very sophisticated argument that there were two different kinds of infinities. I read all of the books on science in the local library and was frustrated when there was nothing left to read.

I grew up in New Jersey and, since there wasn't a state university at the time, I went to the University of Michigan. Since both of my parents were the first generation of people in their families to go to college—my father was an engineer, my mother an artist—there was never a question that I would go to college. I wanted to go to MIT or Cornell, but my parents decided that those institutions were too expensive and the University of Michigan was affordable. I was lucky enough to get into the honors program at Michigan. I had very advanced courses as a freshman and received a superb education. I had a junior-level math course which I found very exciting. I had intended to major in physics and decided to change majors when they started taking attendance in the physics lecture. I also had trouble with labs—I could not learn to look up answers in the back of the book and fudge the experiments. I could never seem to get the labs to come out right. So I switched to math and have been interested in it ever since.

There are three women Ph.D. mathematicians from my freshman honors class at Michigan. Some people at the University of Michigan have a theory to explain this phenomenon of success rates of women from their honors program during this time period: bright women were not sent to expensive, private colleges, so they came to places like Michigan with honors programs. If we had been bright men, they suggest, our fathers would have forked out the money to send us to Ivy League schools.

Karen and John

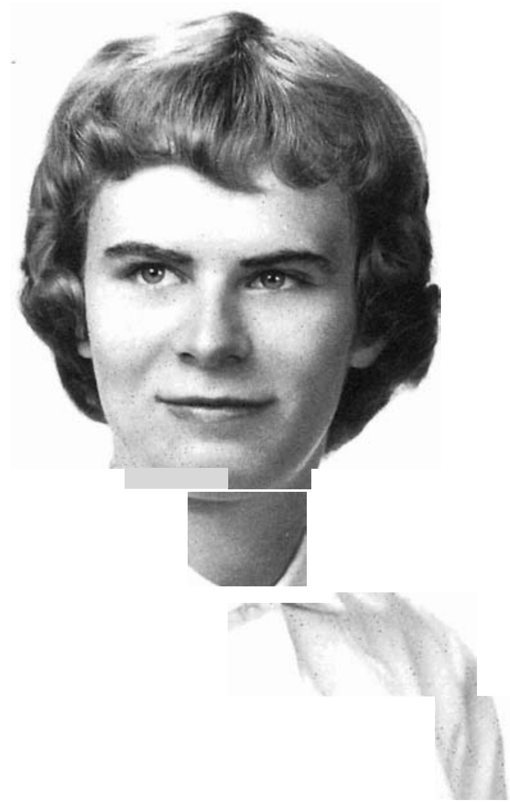
After undergraduate school I spent a year at New York University's Courant

Institute (1964), but then married a biochemist who was going to Harvard, so I switched to Brandeis. I had a National Science Foundation graduate fellowship at that time, so four years of my graduate school were paid for at a very luxurious rate. I was one of the people who benefited from Sputnik. There was a handful of women in my graduate program, although I was not close friends with any of them. It was evident that you wouldn't get ahead in mathematics if you hung around with women. We were told that we couldn't do math because we were women. If anything, there was a tendency to not be friendly with other women. There was blatant, overt discouragement, but also subtle encouragement. A lot of people appreciated good students, male or female, and I was a very good student. I liked doing what I wasn't supposed to do, it was a sort of legitimate rebellion. There were no expectations because we were women, so anything we did well was considered successful.

I have always known that I was a really good mathematician. I have a natural bent for abstraction and I love ideas of all sorts. I value time to be by myself and think, about math or other things, it doesn't matter. The noise of the world is a difficult thing for me to deal with. I have always had a hard time handling external stimuli.

My first husband's parents were older European intellectuals and my father-in-law was a famous physicist. They were very influential in my life. They had a different attitude toward life than Americans. I remember my mother-in-law reading Proust and giving me her English version when she learned to read it in French. My in-laws valued intellectual things in a way that my parents didn't; my parents did value such things, but they believed that making money was more important. I don't think I would have survived at that stage of my career without the encouragement from my first husband's family.

After graduate school I had two temporary jobs. I taught for a year at MIT while my husband was finishing his Ph.D. in biophysics at Harvard, and then I went for two years to the University of California at Berkeley during the Vietnam War. I was not the only woman in those respective departments, and I must say that all of these women (my contemporaries) succeeded spectacularly, probably because they had made up their minds to do what they chose.



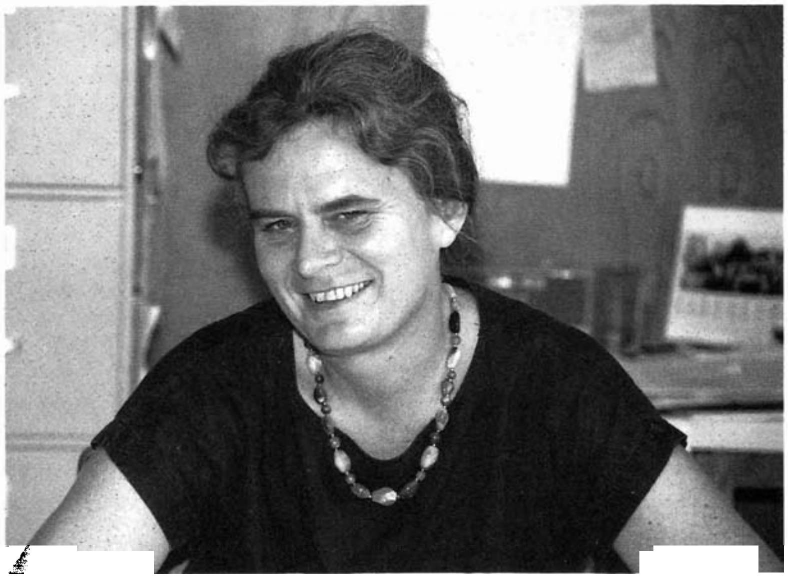
As a high school student, Karen dreamed of a career in science.

I'm still processing a lot of what happened during those years. I think that some of the origin of older women's lack of sympathy with feminists resulted from the fact that many of us were going along fine in our careers, and then somebody started shouting that you were nobody and you weren't supposed to be there. But there you were, and suddenly there was all this fuss about women. Now they had to hire women. It bewildered many of us. It's nice to know that maybe some of the roadblocks have been removed, but I bet that what actually happened was not very useful to anybody.

I was told, when looking for jobs after my year at MIT and two years at Berkeley, that people didn't hire women, that women were supposed to go home and have babies. So the places interested in my husband—MIT, Stanford, and Princeton—were not interested in hiring me. I remember being told that there were nepotism rules and that they couldn't hire me for this reason. When I challenged them on this issue years later, they didn't remember saying these things and, interestingly enough, there were no nepotism rules "on the books." I would have rather they'd been honest and said they wouldn't hire me because I was a woman than lie. Conversely, I would have been just as offended if they'd hired me because I was a woman. I want to be valued for my work as a mathematician, not because I'm a member of a particular group.

At that point in time people were saying all kinds of things about women, most of which had nothing to do with me personally. Prejudice is very rude because it treats you as a member of a class or group instead of as a person. People were tremendously rude.

I ended up at the University of Illinois, Champaign-Urbana because they hired me and my husband. In retrospect I realize how remarkably generous he was because he could have been at MIT, Stanford, or Princeton. I hated Champaign-Urbana—I felt out of place mathematically and socially, and it was ugly, bourgeois and flat. I was lucky to receive a Sloan Fellowship and, instead of doing something mathematically useful, I took time off from teaching to rearrange my life. I had already met Lesley Sibner, who has served me since as role model and advisor for many years. I also started to work with Jonathan Sacks and was taught Teichmueller theory by Bill Abikoff. These were my first close mathematical contacts. I moved to Chicago, established what has proven to be a long-term relationship with Bob Williams (a somewhat older mathematician), taught temporarily at Northwestern and then permanently at the University of Illinois at Chicago Circle. I also became friends with S.T. Yau, whom I credit with generously establishing me finally and definitively as a mathematician.



Uhlenbeck claims that "it's hard to be a role model—you really need to show students how imperfect people can be and still succeed."

I moved from Chicago Circle, with some regrets, to the University of Chicago in 1982, the same year I received a MacArthur Fellowship. It has been a struggle for me to come to grips with my own success. By looking around me at the fate of other women who wanted to be mathematicians, I can intellectually, if not emotionally, understand that this is not so surprising. Not that the fate of other women is surprising, but I really don't understand my success.

I think what has changed today is that people are tremendously more subtle, so that you don't know what it is you're up against. This is true not only for women but for a lot of young people. Young people today are up against the fact that most of the young scientists are coming from abroad, and so most of the people coming into academia are being trained somewhere other than the United States. No one ever talks about this phenomenon of who is actually succeeding in the sciences and engineering—foreign-born men and women. I try to talk about this with my students. It's difficult, however, because you're not supposed to talk about it. In the large classes of engineering students I teach, I'm seeing a lot more diversity—women, Hispanics, African-Americans. It can be done, not just by white, Anglo men.

"My first love is the outdoors."

I am currently at the University of Texas in Austin, and there are three women in the math department, two full professors and one associate professor. I run a mentoring program for women in mathematics which is two years old. I am aware of the fact that I am a role model for young women in mathematics, and that's partly what I'm here for. It's hard to be a role model, however, because what you really need to do is show students how imperfect people can be and still succeed. Everyone knows that if people are smart, funny, pretty, or well-dressed they will succeed. But it's also possible to succeed with all of your imperfections. It took me a long time to realize this in my own life. In this respect, being a role model is a very un-glamorous position, showing people all your bad sides. I may be a wonderful mathematician and famous because of it, but I'm also very human.

Karen Uhlenbeck lives on ten acres in the Hill Country west of Austin with her partner Bob Williams, who is also a mathematician, and three cats, who are not mathematicians.

*Excerpted from **No Universal Constants: Journeys of Women in Science and Engineering**, edited by Ambrose, et al, Temple University Press, to be published in fall 1996.*

Talkative Eve

Talkative Eve

This Cryptarithm (or alphametic, as some puzzlists prefer to call them) is an old one of unknown origin, surely one of the best and, I hope, unfamiliar to most readers:

$$\frac{EVE}{DID} = .TALKTALKTALK\ldots$$

The same letters stand for the same digits, zero included. The fraction EVE/DID has been reduced to its lowest terms. Its decimal form has a repeating period of four digits. The solution is unique. To solve it, recall that the standard way to obtain the simplest fraction equivalent to a decimal of n repeating digits is to put the repeating period over n 9's and reduce the fraction to its lowest terms.

Three Squares

Using only elementary geometry (not even trigonometry), prove that angle C in Figure 1 equals the sum of angles A and B.

I am grateful to Lyber Katz for this charmingly simple problem. He writes that as a child he went to school in Moscow, where the problem was given to his fourth-grade geometry class for extra credit to those who solved it. "The number of blind alleys the problem leads to," he adds, "is extraordinary."

MARTIN GARDNER is best known for his long running "Mathematical Games" column in *Scientific American*. He has published five books with the MAA.

Red, White, and Blue Weights

Problems involving weights and balance scales have been popular during the past few decades. Here is an unusual one invented by Paul Curry, who is well known in conjuring circles as an amateur magician.

You have six weights. One pair is red, one pair white, one pair blue. In each pair one weight is a trifle heavier than the other but otherwise appears to be exactly like its mate. The three heavier weights (one of each color) all weigh the same. This is also true of the three lighter weights.

In two separate weighings on a balance scale, how can you identify which is the heavier weight of each pair?

Answers

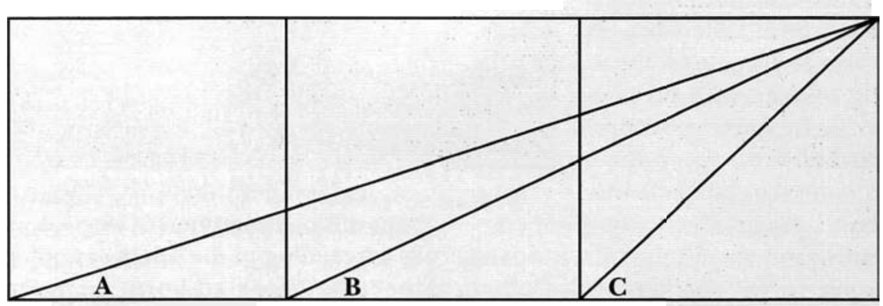
Talkative Eve

As stated earlier, to obtain the simplest fraction equal to a decimal of n repeated digits, put the repeating period over n 9's and reduce to its lowest terms. In this instance TALK/9,999, reduced to its lowest terms, must equal EVE/DID. DID, consequently, is a factor of 9,999. Only three such factors fit DID: 101, 303, 909.

If DID = 101, then EVE/101 = TALK/9,999, and EVE = TALK/99. Rearranging terms, TALK=(99) (EVE). EVE cannot be 101 (since we have assumed 101 to be DID) and anything larger than 101, when multiplied by 99, has a five-digit product. And so DID=101 is ruled out.

If DID=909, then EVE/909=TALK/9,999, and EVE=TALK/11. Rearranging terms, TALK = (11) (EVE). In that case the last digit of TALK would have to be E. Since it is not E, 909 also is ruled out.

Only 303 remains as a possibility for DID. Because EVE must be smaller than 303, E is 1 or 2. Of the 14 possibilities (121, 141,...,292) only 242 produces a decimal fitting .TALK-TALK..., in which all the digits differ from those in EVE and DID.



Prove that angle A *plus angle* B *equals angle* C.

Figure 1

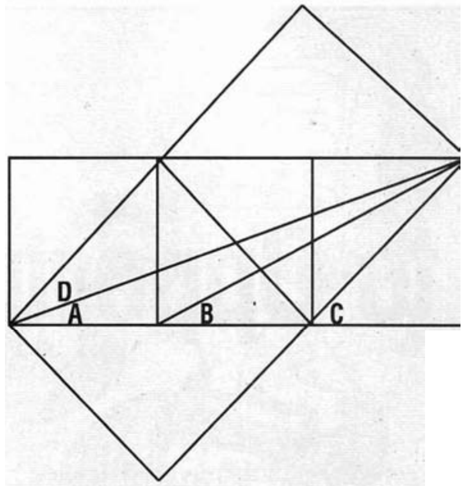
The unique answer is 242/303=.798679867986...If EVE/DID is not assumed to be in lowest terms, there is one other solution, 212/606=.349834983498..., proving as Joseph Madachy has remarked, that EVE double-talked.

Three Squares

There are many ways to prove that angle *C* in the figure is the sum of angles *A* and *B*. Here is one [see figure 2]. Construct the squares indicated by gray lines. Angle *B* equals angle *D* because they are corresponding angels of similar right triangles. Since angles *A* and *D* add to angle *C*, *B* can be substituted for *D*, and it follows immediately that *C* is the sum of *A* and *B*.

This little problem produced a flood of letters from readers who sent dozens of other proofs. Scores of correspondents avoided construction lines by making the diagonals equal to the square roots of 2, 5, and 10, then using ratios to find two similar triangles from which the desired proof would follow. Others generalized the problem in unusual ways.

Charles Trigg published 54 different proofs in the *Journal of Recreational Mathematics*, Vol. 4, April 1971, pages 90–99. A proof using paper cutting, by Ali R. Amir-Moéz, appeared in the same journal, Vol. 5, Winter 1973, pages 8–9. For other proofs, see Roger North's contribution to *The Mathematical Gazette*, December 1973, pages 334–36, and its continuation in the same journal, October 1974, pages 212–15. For a generalization of the problem to a row of *n* squares, see Trigg's "Geometrical Proof of a Result of Lehmer's," in *The Fibonacci Quarterly*, vol. 11, December 1973, pages 539–40.



Construction for proof of three-square theorem

Figure 2

Red, White and Blue Weights

One way to solve the problem of six weights—two red, two white, and two blue—is first to balance a red and a white weight against a blue and a white weight.

If the scales balance, you know there are a heavy and a light weight on each pan. Remove both colored weights, leaving the white weights, one on each side. This establishes which white weight is the heavier. At the same time it tells you which of the other two weights used before (one red, one blue) is heavy and which is light. This in turn tells you which is heavy and which is light in the red-blue pair not yet used.

If the scales do not balance on the first weighing, you know that the white weight on the side that went down must be the heavier of the two whites, but you are still in the dark about the red and blue. Weigh the original red against the mate of the original blue (or the original blue against the mate of the original red). As C.B. Chandler (who sent this simple solution) put it, the result of the second weighing, plus the memory of which side was heavier in the first weighing, is now sufficient to identify the six weights.

For readers who liked working on this problem, Ben Braude, a New York City dentist and amateur magician, devised the following variation. The six weights are alike in all respects (including color) except that three are heavy and three light. The heavy weights weigh the same and the light weights weigh the same. Identify each in three separate weighings on a balance scale. ■

Gardner welcomes your comments, problems, and solutions. Write to him at the following address:
Martin Gardner, 3001 Chestnut Road, Hendersonville, NC 28792.

DUANE A. COOPER

A Gateway to Opportunities

What can you do with a math major, besides teach? One answer is to make good use of your strong mathematical foundation as you attain an advanced degree in another discipline that interests you.

Options Galore

Those of us who have taught and advised college mathematics students are familiar with the "What can I do . . . ?" query. I will ignore, here, the often implicit idea that teaching mathematics is a bad thing. Instead, I will address the notion that the pursuit of mathematics is limiting, a view that inhibits many talented math students from fulfilling their mathematical potentials. As a dismaying consequence, the pool of strong math majors—some of whom would make contributions to mathematics beyond college shrinks as students who like math and are good at math flee for majors in business, engineering, and the like, which they perceive as more marketable.

The Mathematical Association of America (MAA) is publishing valuable literature on career opportunities—beyond teaching—for math majors. *Math Horizons* articles, such as "How to Really Get a Job" (September, 1994) by Anita Solow, highlight an assortment of career paths taken by recent math graduates. MAA pamphlets and booklets, (Careers in the Mathematical Sciences; More Careers in the Mathematical Sciences, *Mathematical Scientists at Work*) offer profiles of former math majors working in a wide variety of professions.

The Solow article cites a math graduate in the workforce who advises students to "stay in school and get a master's degree," similar to this author's frequent recommendation to continue or return later for a master's, at least. It is important to realize, though, that a math major's advanced study is not limited to a choice of mathematics, statistics, or education. Students often do not know that a math major does not wed them to mathematics for life.

People change fields all the time. Graduate students often come to their departments from different undergraduate disciplines. Professional schools do not prescribe the commonly studied college majors of business or economics for business school, biology or chemistry for medical or dental school, and political science for law school. With appropriate backgrounds, a host of graduate and professional programs welcome students who have earned bachelors degrees in mathematics.

Questions

A brief, two-item questionnaire was sent to a wide range of graduate and professional schools at the University of California, Berkeley, and the University of Maryland, College Park—a non-scientific poll conducted at two state institutions. Almost all of the graduate programs surveyed are Ph.D.-granting departments, though graduate study was not specified as being for a master's or a doctorate. Professional programs at the University of California, San Francisco, and the University of Maryland, Baltimore, were also polled. Replies were plentiful—46 of the 84 recipients (55%) mostly from professors, with others from persons serving in administrative capacities. Responses were requested to the following questions:

1. Do you/would you accept math majors into your graduate school or professional program? If so, what courses or training (if any) should the applicant have besides his/her mathematics background?

2. What advantages (if any) might a mathematics major have as a student in your graduate program or professional school? What disadvantages ... ?

and Answers

The results show that most of the advanced degree programs would indeed accept math majors ["I would cherish the opportunity to have mathematics undergraduates as graduate students in my lab."] provided that they possess sufficient knowledge of the proposed new discipline. To some respondents, that "sufficient knowledge" meant having taken many of the courses required of undergraduate majors in their departments. ["Two years of chemistry and introductory biological sciences would be expected." "A non-trivial amount of either electrical engineering or computer science courses."] Other departments would require only some core courses in their disciplines

DUANE A. COOPER is an assistant professor in the Center for Mathematics Education and the Department of Mathematics at the University of Maryland at College Park. He used his undergraduate math major at Morehouse College as a gateway to a master's degree in electrical engineering at Georgia Tech.

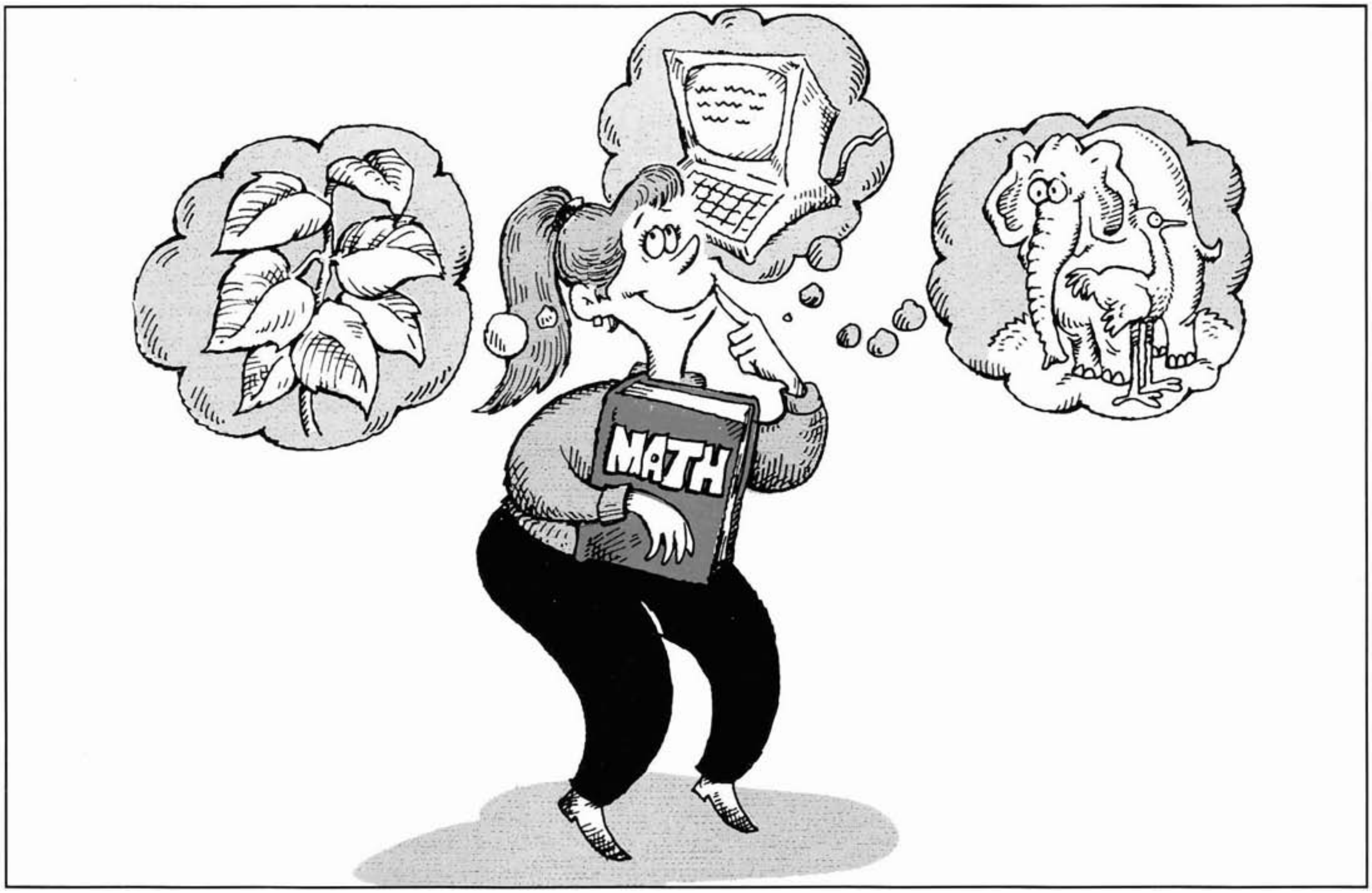


Illustration by Loel Barr

and a sincere demonstration of desire to pursue the new field of study. ["Some philosophy courses, not much mattering which." "Some indication, either through course work or work experience, of a commitment to and interest in public service."] A double major, of course, is a means of satisfying the requirements of a math major and of a graduate department in a second discipline, but not every student's schedule or desire accommodates this.

Some respondents said, 'no,' they would not accept math majors. ["Unlikely. Too many qualified astronomy applicants."] Their replies and explanations, too, are included in the "Excerpts" section at this article's end. A University of Maryland physics professor disagreed with my premise entirely; "I do not believe pure math is a suitable 'gateway' in today's job climate. You should encourage double majors or major/minor combinations."

Advantages commonly cited were math majors' analytical thinking, reasoning abilities, and capacities to model situations mathematically. ["Ability to deal with formal systems and to think abstractly." "Being comfortable proving mathematical theorems and doing algebraic manipulations."] Some disciplines have areas that are particularly theoretical or otherwise advanced mathematically. ["There is an entire range of theoretical problems which can only be studied by students with a good background in mathematics."] Other fields have courses and research areas that are quite quantitative. ["Entomology is becoming more quantitative. The skills learned by math majors would greatly help students in understanding current literature and in using quantitative approaches in their own research."] The math major's undergraduate training can prove quite valuable in these cases.

The disadvantage most frequently mentioned was that—except for students with double majors—the undergraduate math major could be hindered by deficiencies in his/her background in the new field of study. ["Generally poorer preparation in undergraduate engineering subjects." "Unfamiliarity with the basic literature of the social sciences."]

The results of this account should encourage a student who likes mathematics and is good at mathematics to continue to study mathematics. A math major who has or develops an interest in studying another subject at the graduate level should investigate available opportunities. The student can inquire at his/her institution or elsewhere to get the opinions of professors and admissions officers other than those quoted in the excerpts below.

Some Excerpts

Astronomy. "[A math major needs] a thorough undergraduate background in physics including quantum mechanics, classical dynamics, statistical physics and thermodynamics, and electromagnetism. A lab course would be helpful... We occasionally take math students without this much physics, but then they need to do remedial work." Jonathan Arons, Chairman, Dept. of Astronomy, UC Berkeley.

"Unlikely. Too many qualified astronomy applicants [An advantage would be that] mathematics is a vital tool in all that we do in astronomy. An undergraduate physics and astronomy background seems essential for a successful graduate career." Marvin Leventhal, Chairman, Dept. of Astronomy, UM College Park.

Botany. "Yes, [but they need] physics, chemistry, and biology—at least two semesters worth of all three subjects. [Advantage:] Much greater facility to quantify the phenomena of research interest. [Disadvantage:] Very limited biological intuition.

"I work on the plant morphogenesis, which attempts to characterize how plants generate their characteristic shapes. I would cherish the opportunity to have mathematics undergraduates as graduate students in my lab." Todd Cooke, Director of Graduate Studies, Dept. of Botany, UM College Park.

Business and Management. "Math graduates do fine in our MBA program. The program is fairly quantitative and in some cases, math majors may have an advantage. Math majors do especially well in research-oriented areas like operations management and information systems." Hayden Estrada, Director of MBA/MS Admissions, College of Business & Management, UM College Park.

Chemical Engineering. "Most likely not ... unless there is strong interest in applications as indicated by significant course work in applied science/engineering." Simon Goren, Chair, Dept. of Chemical Engineering, UC Berkeley.

"Yes. [They would be] required to take a few core chemical engineering undergraduate courses. . . There are some research areas in chemical engineering where advanced math plays an important role. Students with math background, by adding fundamental chemical engineering knowledge, can develop into fine chemical engineers." K. Y. Choi, Graduate Program Committee Chair, Dept. of Chemical Engineering, UM College Park.

Chemistry. "Yes, but they must also have had the required prerequisite courses—e.g., freshman chemistry, organic chemistry, physical chemistry...I am of the opinion that you can never have too much math.

"The physical chemistry and nuclear chemistry and even some [other] disciplines have specialties which are mathematically intensive, and having been a mathematics major may be a distinct advantage.... Your average math major might find experimental work a bit foreign." Alice C. Mignerey, Associate Chair for Graduate Studies, Dept. of Chemistry and Biochemistry, UM College Park.

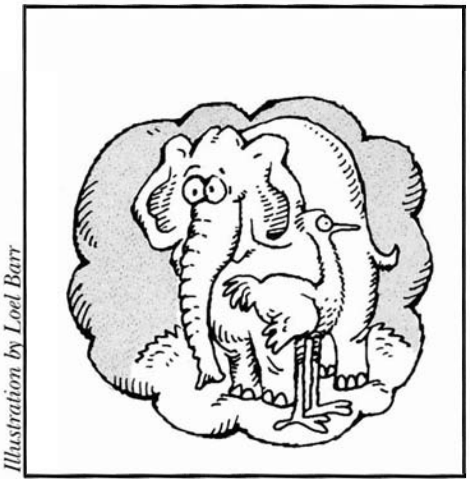


Illustration by Loel Barr

City and Regional Planning. "We do accept math majors into the [masters] program. We like for all our students to have at least one course [each] in . . . basic statistics, microeconomics, and introductory political science. We would also like for our students to be computer-literate, not so much with programming, . . . but with spreadsheets, databases, and word processing programs.

"The playing field is becoming increasingly quantitative, and these skills are extremely valuable, whether you're an urban designer or an economic development specialist or a land or real estate developer. . . . Writing is extremely important in this field, and these skills should be cultivated." Kaye Bock, Graduate Assistant Supervisor [in charge of admissions and student affairs], Dept. of City & Regional Planning, UC, Berkeley.

Computer Science. "Math majors are okay . . if they have taken a substantial number of c.s. courses and acquired a demonstrable proficiency (by doing well on the C.S. GRE, for instance). Advantages: Good analytical strengths; helpful for work in theory or areas like robotics/vision. . . . Disadvantages: Will have to work harder initially to make up deficiencies in course work; may not have enough programming/hardware experience." Jitendra Malik, Admissions Committee, Computer Science Division, UC Berkeley.

Dentistry. "Yes, we do currently accept math majors. Minimal science prerequisites for dental school include one year of inorganic chemistry and one year of general biology," Margaret B. Wilson, Assistant Dean for Admissions & Student Affairs, School of Dentistry, UM Baltimore.

Electrical Engineering. "UC Berkeley does, on occasion, accept math majors into electrical engineering and computer science graduate programs... It is to the applicant's advantage to have taken a non-trivial amount of either electrical engineering or computer science courses as an undergraduate." Mary Byrnes, Student Affairs Office, Dept. of Electrical Engineering & Computer Science, UC Berkeley.

Entomology. "We would accept math majors, and in some subdivisions of entomology, we would welcome them. However, our students must have a solid background in biology as well, and chemistry is strongly recommended.

"Entomology is becoming more quantitative. The skills learned by math majors would greatly help students in understanding current literature and in using quantitative approaches in their own research." William O. Camp, Graduate Director, Dept of Entomology, UM College Park.

Geology and Geophysics. "Yes. Physics courses, both lower-division and upper-division, are needed. Students cannot take too much math...There is an entire range of theoretical problems which can only be studied by students with a

Illustration by Loel Barr

good background in mathematics." Lane Johnson, Professor, Dept. of Geology and Geophysics, UC Berkeley.

Government and Politics. "Yes, If a student was particularly interested in ... theory. Some work in the social sciences [would be required]....Advantages: Statistics is required; work in formal modeling; work in economics. Disadvantage: Unfamiliarity with the basic literature of the social sciences or of political science." Don Piper, Director of Graduate Studies, Dept. of Government & Politics, UM College Park.

Industrial Engineering and Operations Research. "Yes, we do admit math majors. The math majors in our program have the advantage of being comfortable proving mathematical theorems and doing algebraic manipulations...A few...have difficulty using mathematics to model real-world phenomena...and difficulty developing intuition for normative (vs. descriptive) models with economic implications." Candace Yano, Chair, Graduate Admissions, Dept. of Industrial Engineering and Operations Research, UC Berkeley.

Journalism. "We would accept qualified applicants who majored in math, but few, if any, apply. We expect journalists to be sensitive to language and to organize their thoughts clearly and coherently. This is a field that highly values people who are enthusiastic, creative, and who demonstrate initiative. Undergraduate coursework and outside experiences that nurture these skills are important for all applicants.. . . We require students to produce a tremendous volume of clear writing." Margaret Miller, Academic Coordinator, Graduate School of Journalism, UC Berkeley.

Law. "Some 60 applicants who majored in mathematics applied for the 1994 entering class; ....10 were admitted (20%). Given the overall ratio ... is 14%, it appears that math majors do well in our process. The Admissions Committee seeks strong academic potential combined with analytical skills.... Successful applicants also augment their curricula with writing courses and other liberal arts courses." Edward G. Torn, Director of Admissions & Financial Aid, School of Law (Boalt Hall), UC Berkeley.

Linguistics. "We have accepted math majors—and I think most other major linguistics departments have too....Several major linguists I know were math majors...We like our graduate students to have high math ability ... and we find that math background is a plus.

"Anyone trying to get admitted to a linguistics program should have some background in linguistics—preferably a coherent grouping of undergraduate courses, if not a major. . . . Experience with natural languages is a real plus, almost a necessity: students should know at least one other language well besides their native language and should have thought about its structure.

"I can't think of any disadvantages to a math major. A major advantage is that math majors are not scared of formal models and also don't confuse formalizing a problem with solving it." Eve Sweetser, Chair of Graduate Studies Committee, Linguistics Dept., UC Berkeley.

Medicine. "Applicants of all majors and academic backgrounds are considered for medical school provided they have done well in their chosen field. They must take a "core" of courses...two semesters each of English, biology, general chemistry, organic chemistry, physics...[We] feel that good performance in math courses indicates adequate analytical skills...Students are encouraged to pursue activities outside of the classroom or lab." Milford Foxwell, Dean, Office of Admissions, School of Medicine, UM Baltimore.

Microbiology. "Yes, we would accept math majors into our graduate program provided that they present evidence supporting an interest in microbiology....Two years of chemistry and introductory biological sciences would be expected....The disadvantage for most would be the omission of background courses and lab experiences that most graduate students would have had. Some of these . . . would have to be remedied. For others, laboratory sophistication obtained from job experiences, etc, could be adequate." Bob S. Roberson, Professor, Dept. of Microbiology, UM College Park.

Music. "We require our graduate students to have a background in theory, musicianship, and music history. . . . As long as students have the necessary skills, it doesn't matter whether they are music majors.

"There certainly seems to be a connection between musical and mathematical aptitude, but I can't think of any direct advantages math majors would have." Bruce Alexander, Student Affairs Officer, Dept. of Music, UC Berkeley.

Nuclear Engineering. "We have [accepted] and continue to accept math majors. Entering students should have the [appropriate] background [in] physics, chemistry, thermodynamics, fluid flow, and interaction of radiation with matter. Students lacking portions of this material may make it up as a graduate student. Advantage: Ability to deal comfortably with partial differential equations, especially [in] reactor physics. Disadvantage: Generally poorer preparation in undergraduate engineering subjects on which graduate course work is based." Don Olander, Professor (in charge of Graduate Admissions), Dept. of Nuclear Engineering, UC Berkeley.

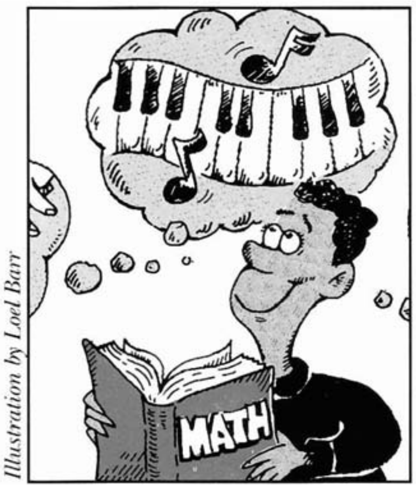


Illustration by Loel Barr

Nutritional Sciences. "No. [We] would require several years training in chemistry, biochemistry, and biology. Also, a year of experience in a biology/biochemistry laboratory is usually required." Len Bjeldanes, Professor, Dept. of Nutritional Sciences, UC Berkeley

Optometry. "Of course! We have no required major...but many prerequisites: general chemistry, organic chemistry, biochemistry, general biology or zoology, calculus/mathematics, general physics, reading & composition, microbiology, comparative or human anatomy, human physiology, statistics, and psychology." Sandy Jaeger, Student Affairs Officer, School of Optometry, UC Berkeley.

Philosophy. "We do, and we would [accept math majors]. [The applicant should have] some philosophy courses, not much mattering which.... Philosophy requires orderly thinking and reasoning ability, both of which are common to math majors.... [A math major may have] not much experience with evaluative, moral, social, and aesthetic questions, [nor] acquaintance, with history of philosophy." Jerrold Levinson, Director of Graduate Studies, Dept. of Philosophy, UM College Park.

Physics. "We would not accept any student who had not had most of the physics courses of a physics major, though a strong electrical engineering background might suffice... A double math/physics major has a real advantage in theory courses, which dominate the first two years of graduate school." Richard Ellis, Associate Chair for Graduate Education, Dept. of Physics, UM College Park.

Policy Studies. "Yes ... as long as there is some indication, either through course work or work experience, of a commitment to and interest in public service.... Since we are a quantitative program, math ability is extremely useful in the first year/core curriculum." Elisse Y. Wright, Assistant Dean, School of Public Affairs, UC Berkeley.

Psychology. "We definitely would accept math majors. Approximately 16 to 21 psychology credits would be needed.... A mathematics background is excellent for industrial psychology. . . . Our students need a strong background in psychology or research experience...The math major should be helpful in the required statistics courses and for data analysis." Betty Padgett, Dept. of Psychology, UM College Park.

Zoology. "We require one year [each of] introductory chemistry, organic chemistry, physics, and calculus, [and the] Biology GRE. Linear algebra and statistics are very useful to graduate students who study population biology. . . . Several applied math students have pursued research in our department in either computational neuroscience or theoretical population genetics. However, these students often suffer from inadequate biology background to permit them to generalize their models." Jerry Wilkinson, Director of Graduate Studies, Dept. of Zoology, UM College Park.

ROBERT C. ESLINGER AND JON W. SCOTT

Life After Calculus

Bridging the Gap between Beginning and Advanced Mathematics

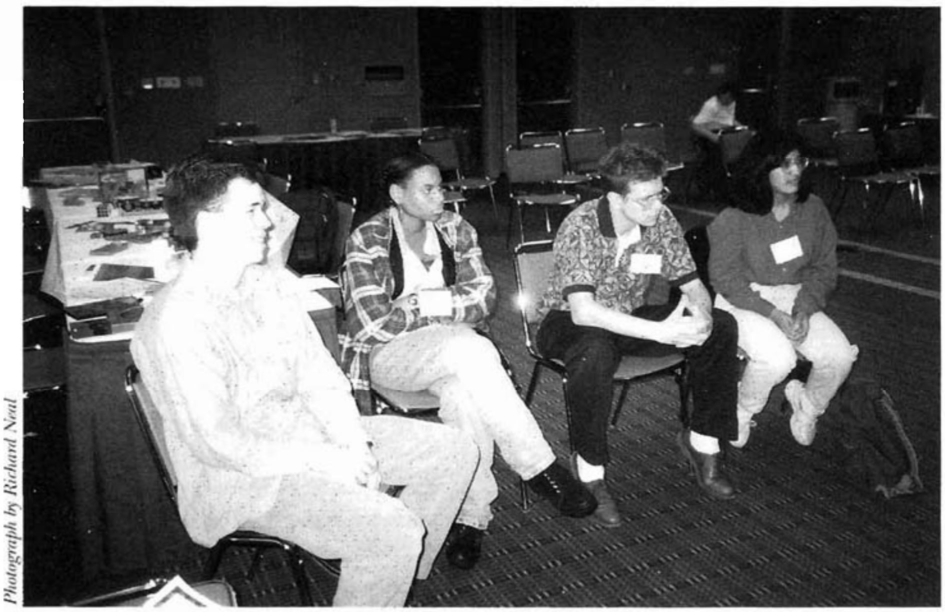
The setting is the Student Hospitality Center at this winter's Joint Mathematics Meetings in Orlando, Florida. In the midst of a busy and productive meeting (not to mention the "Blizzard of '96"), the authors, both MAA Visiting Mathematicians, spent a delightful couple of hours talking with students about making the transition from lower to upper level mathematics courses. Let's listen in on the conversation.

How are upper level mathematics classes different from calculus? We define calculus broadly to include multivariate calculus and differential equations.

Dan Isaksen *Second year graduate student University of Chicago* The main difference is the proof writing. It is the big thing. The emphasis changes from doing calculations to proving theorems.

Darren Glass *Junior Rice University* Another difference is that in calculus classes there are a lot of people who are going on into other things—engineering, biology, business—so that math is not their main interest. They don't care as much about it. And the classes are bigger. The upper level classes normally are smaller, and most of the people are there because they want to be, because they are interested in the subject. Having smaller classes allows teachers to be more flexible, more willing to work with you one-on-one.

ROBERT C. ESLINGER is professor of mathematics at Hendrix College and a visiting mathematician at the MAA for the 1995-96 school year. **JON W. SCOTT** is professor of mathematics at Montgomery College and a visiting mathematician at the MAA for the 1995-96 school year.



Photograph by Richard Neal

Tom Travison, Ayana Moore, Ryan Siders and Ruth Britto-Pacumio.

Izzy Kyle *Junior Hood College* We have been doing a lot of group projects, which is a lot different from the freshman and sophomore years. There are more research type things and presentations to the class.

Tom Travison *Junior Skidmore College* I have had only one instructor who actually encouraged group presentations in class, but I found that I value more the group discussions outside of class, with four or five people hanging out most of the semester talking about math. I think students learn a lot from each other. The more group discussion, the better.

Ryan Siders *Senior University of Minnesota* The most striking feature to me is the pace at which ideas come to you. In the first year, there are maybe two ideas—integration and differentiation. But compare that to the first algebra or topology course—so many ideas and so many shapes and so many structures. Much, much more comes out of this. I find that exhilarating.

I want to go back to Darren's comment that calculus classes have a lot more students from other fields. That's true, but I'm not sure if that is good or bad. I took a course on chaos and a significant percentage of the students were not mathematicians. A lot were liberal arts majors and this was their only math course. One told me that she didn't have to take the course to satisfy her requirements, but felt that it would be a good experience to actually learn math. When I asked her why she didn't take calculus, she surprised me with her answer that she wanted to take a real math course, something rigorous, not "babied down" like she perceived calculus to be.

Ryan Siders

Ruth Britto-Pacumio
Senior
Massachusetts Institute of Technology

There might be a difference between applied mathematicians and career mathematicians. People who are applied math majors love differential equations so much that they decide that is what they want to do for the rest of the time. So for them—I don't really know since I haven't taken the applied math courses—maybe that is not so much of a transition. The pure mathematicians know that they are interested in pure math before they even take calculus and they think of calculus as something else. It is not really part of what they are interested about in math. So they probably don't really think of it as a transition either. They already know what they want to study. They might not know anything about it, but they are sort of interested in learning about algebra or topology, for example, and they know that is going to come later. In the meantime, they are just waiting and taking these required courses.

Ruth Britto-Pacumio

Ayana Moore
Junior
Spelman College

One of the big changes in going from calculus to more advanced classes is just basic maturity. My experience in calculus is that you do what you have to turn in—projects you have to turn in, but as far as studying, you cram everything into two days before the test and you take the test and you get an A. But in more advanced classes, you can't pull that off. It doesn't work. It is basic maturity. You have to study as you go.

Dan: Another transition that I have been through recently from undergraduate to graduate school, and in a lot of ways there are similarities, is that at each stage you go up you have to take more responsibility for your own education—maturity, as Ayana pointed out. In calculus you can really depend on the professor. Go to classes and they tell you everything that you need to know and you can go and take the exam. Perhaps when you get to upper division, you start to need to read the book and there is too much material to cover all in class, so you are on your own.

Ayana: In higher level classes the teacher expects more of an effort, you have to read the books. One of my professors, if you ask a question, will say "did you read?" That was her favorite question. Teachers expect you to take more of an effort on your own. You can't do it all in the classroom. It is not going to happen.

Tom: My linear algebra class was taught in a way that demanded a lot from the class in terms of basic knowledge. Our professor would always answer your questions, but she didn't spend a lot of time going over stuff that was explained in detail in the text. So it forced me to study in a way that I hadn't before. I always felt that I could just pick it up in class and then go over my notebook. I only used the book for homework assignments. So my linear algebra text, which I still have, is the first highlighted book I have. It has comments in the margin. I never did that before in my other books.

Tom Travison

Did many of you have any emphasis in your early math courses on theorem proving? Was it a big shock to you when you got into the upper level classes? Were you prepared for it? Was it expected?

Izzy: Well, at Hood, the program is small and the professors and the students all know each other. The professors know your background and work with that. Going into a proof-writing course, most of the class wasn't familiar with it or adept at doing it, but the teacher never catered to that. I think to everyone it is an initial shock.

Izzy Kyle

Darren: I remember that my high school teacher and my calculus teacher would always say, "Well, pretty soon you are going to have to start working on proofs." So it wasn't a shock. I knew it was coming, but that didn't make it any easier when it did come.

Dan: I remember my first course where I was expected to prove theorems. They started giving homework that was proving theorems and I just started doing it. I did my best and tried and got comments from the homework grade and so forth. I just went along and throughout the semester, without any particular program or design, I learned how to write theorems, just by doing it. I jumped into the deep end and stayed afloat

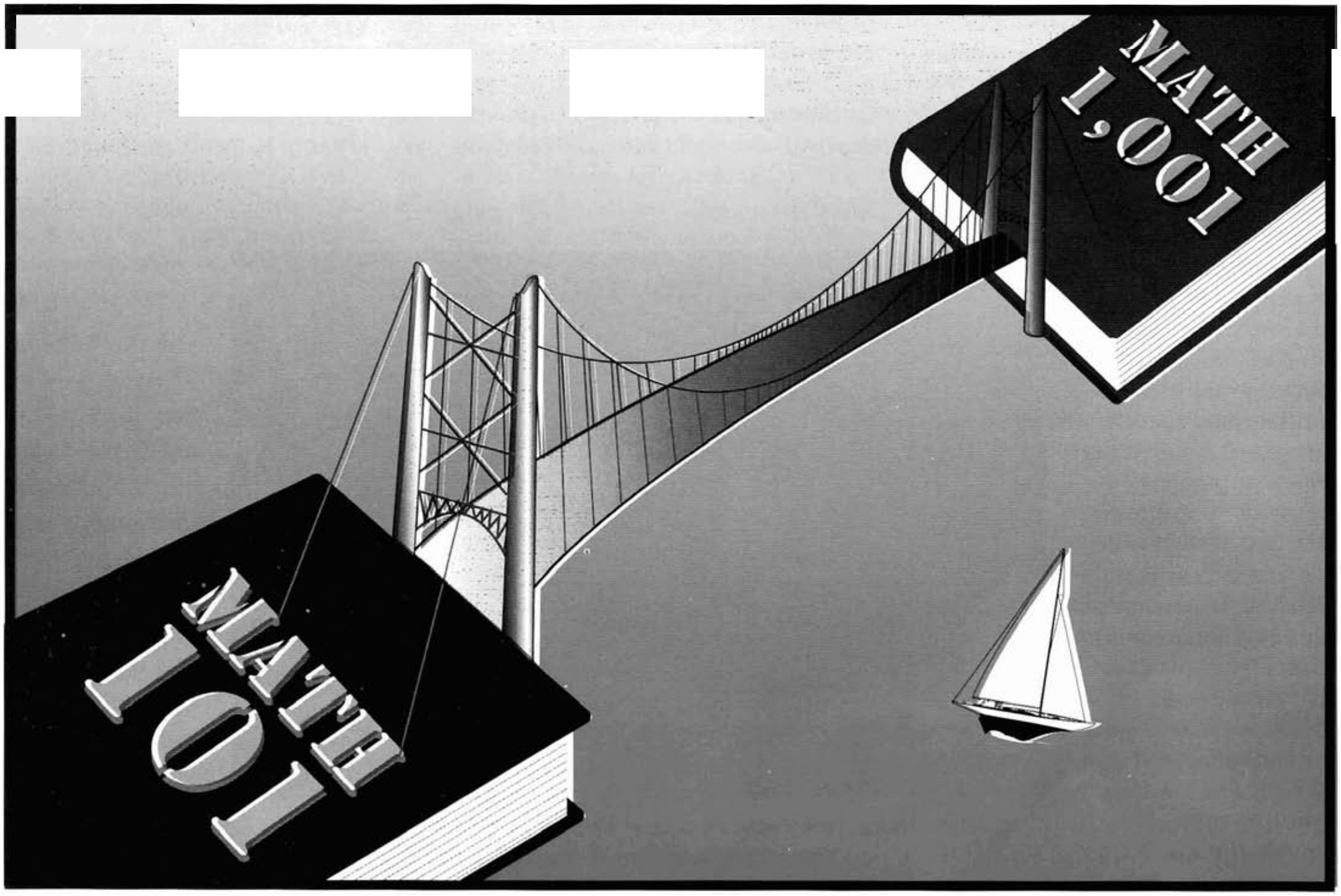


Illustration by Marty Tatum of Ice House Graphics

somehow or other. I went back recently and was looking through my old homework that I had kept. I was sort of shocked at the quality of what I had done. But I could see during that first semester a big change. By semester's end I had learned just by doing it. It just happened by practice.

Ruth: My first experience with proofs was in writing two-column proofs in high school geometry. The transition is going from two-column proofs to paragraph proofs, but I think the basic concepts were built into the two-column proofs.

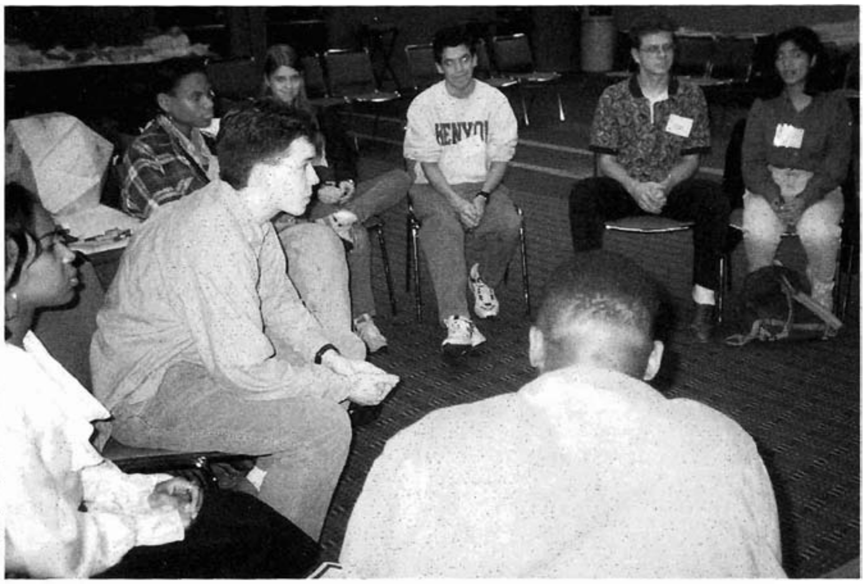
Dan: I worked one summer with a program for high school students where they were learning some elementary number theory. A big part of it was teaching the students how to write proofs. We would start off with two-column proofs; they had perhaps seen that in geometry and they could understand statement, reason, statement, reason, and so forth. Then we sort of worked them into rewriting these two-column things into paragraphs. Just add some English between statements and reasons and put in punctuation. I think in a lot of ways it is the same. The format is different, but the underlying principle really is the same.

Ayana: I guess you can run ahead of yourself in proofs and get stuck. You just need to back up and slow down. You have to go back to two-column proofs because they are very basic step-by-step procedures. You have to ask yourself where can I move from here. If you go back to writing down statements and reasons, you can eventually reason it out. I think that helps a lot.

Did any of you have calculus taught in a nontraditional manner where you had a laboratory component, made use of technology, or did a lot of writing? Did that experience help provide a bridge between your lower and upper level classes?

Izzy: We worked with a program called Project CALC which is far from traditional. Calculus for us was a laboratory course and once a week, just like in biology or chemistry, we had a lab. It was actually quite theoretical. You would learn about population growth or something and then, two hours later, would realize, "Oh look, I'm learning about exponentials!" It was a lot of learning by discovery, and I think in that way my transition to upper level classes was a lot easier because explaining myself and discovering things for myself was something I became used to.

I had regular calculus in high school and when I got to college I was expecting to have an easy time with it, but it was totally different. I think it frustrates some people because they are expecting a regular textbook, work 50 problems, and hand it in. Just boom, bang, get it done. A lot of people who consider themselves good at that kind of thing are sometimes frustrated because they are challenged so much in our calculus program because it is so rigorous. You are really pushed to think and explain yourself. On the exams there are some conventional "plug and chug", but there also are questions requiring more explanation, and just writing some fluff isn't going to cut it. A lot of people think it is frustrating, but for me, continuing my math education made the transition a lot smoother.


Photograph by Richard Neal

Pamela Moses, Tom Travison, Ayana Moore, Karen Downey, Tim Bator, Ryan Siders, and Ruth Britto-Pacumio.

Ayana: I had a strong calculus course in high school, so when I came to college, I took calculus and it wasn't bad. But when I got to abstract algebra, it was totally different. There were times when I would stay up all night getting nothing done at all. My roommate would tell me to go to bed, and I would say that I had to get at least one problem. But sometimes you can't find an answer. In calculus, you can always find the answer; just sit there long enough. But it was different in abstract algebra. Sometimes, I just couldn't get the answer. I would have to ask for some help from a faculty member or work with someone else. And it wasn't easy. I was so frustrated that I talked about changing my major. But by the end of the semester I had learned I had to adjust to the type of class and to the frustration. I had to learn to hang it up for the night. With calculus, I would sit there until all 50 problems were done and all 50 would get done. But I can't always do all 10 of those abstract algebra problems. It is frustrating, but you just have to learn to deal with that frustration and once you get over that hump, you know you just can't always do everything right there and then.

Tom: Like you were saying, you have to learn to just be able to put it away for awhile. And then the weirdest thing is that you will be in English class or in the shower and you will say "Oh!" and wish you had a notebook. I think the earlier you learn to think independently and on a conceptual level, the easier the transition will be.

Darren: Another difference between hard calculus problems and hard problems from other areas is that if there is a calculus problem that you can't get no matter how hard you stare at it or whatever, it is just really hard, the odds are that ninety percent of the class is sitting there staring at the same problem and not being able to get it. While in an abstract algebra class, if there is a problem that you really can't get, there is a really good chance that the guy sitting next to you in class can get that one, but can't get the other one that you can get, just because you are seeing things slightly differently. So I think that is why it really helps to exchange ideas and to get together after class.

What advice do you have for instructors who are designing and teaching the curriculum and for students who are freshman and sophomores thinking about being math majors that would help make the transition to upper level courses go more smoothly?

Ryan: I think, in my case in particular, long before I had studied calculus, I knew I wanted to be a math major because I had seen interesting problems, like developing a strategy to win at checkers—something like that. In college, I took the four hardest courses in math because I was so interested in these things. I had a blast. I don't think anything should get in the way of that sort of personal drive. I think the job of the teacher in the classroom is to provide an environment in which one can learn a lot of material and which is not available elsewhere. I think the reason you go to college is because it is a place where you can learn more mathematics than you could on your own. The teacher must provide an environment where this is possible.

Ruth: I like Ryan's point that interest provides excellent preparation. A lot of math majors I meet know they want to be math majors before they come to college and it is certainly not calculus that makes that decision for them. They are going to go through calculus and then, even if it is a bit of a shock when they come to these more advanced courses, they are not going to say "I have to prove things now, maybe I should switch fields." They are going to continue to do it and they are going to learn to prove things in the course of the classes that they are taking.

Pamela Moses
Senior
Morris Brown College
It is important to understand what you are doing and why you are doing it as opposed to just going through the problems and saying, "Okay, now I found the answer, let me go on to the next one." So, it's a good idea to outline a chapter, writing down the concept and an example of a problem that you worked through that goes along with that idea. Then you have something to relate to when the idea comes up again.

Tom: I have become really frustrated with students who say things like "I can't do math." Many students who aren't that involved in mathematics seem to think that math is about crunching numbers. I try to tell them it is not and that it is really about the love of thinking. It is very scholarly. These people don't seem to buy it. So I guess my advice for students is just not to be discouraged by that. For instructors, if you are a teacher in any discipline, there is a reason for that and you obviously have some sort of enthusiasm or some sort of love of the subject matter. I think the greatest gift an instructor can give to the student, especially in an area like math, is communication of that enthusiasm.

Ayana: I think it is important to separate those people who are taking math, not only because they are math majors, but because they like it, from those who are just trying to satisfy their math requirement. It should be taught differently to prepare me for moving on to courses like abstract algebra or advanced calculus.

Dan: I agree. An engineer really needs to know how to integrate the inverse trigonometric functions. If they are going to build a bridge, they need to be able to make a calculation and come up with an answer. These things are taught in calculus because they are so important. But mathematicians have a different emphasis. It is nice to know these ideas, but there is a different point to it. I do think it is also nice to teach engineers to think mathematically and logically, but there is a different emphasis and I think it is very hard to combine both goals successfully in a single course.

Teena Conklin
Sophomore
Kenyon College
I don't know if separating classes is a good idea. I learned a lot being in a class with people who were not as mathematically minded as I. In my calculus class, I helped a biology major who thought about things completely differently than I did. She helped me see things in a more global, more interesting way. Her point of view added as much to me as mine did to her.

Karen Downey
Sophomore
Kenyon College
I think our introductory proofs course was extremely helpful in learning to think mathematically. But aside from that, it is important for the instructor to have an open door policy where somebody can just come in and say "I'm confused. Do I have the right approach? Am I thinking about the problem clearly?"

Frederick Rege
Senior
Morris Brown College

Frederick Rege

If you know you are a math major and you are taking these calculus courses, you do the proofs whether they are exciting or not. Sometimes, after the statement of a theorem, books will say "the proof is an exercise for the reader." Well, do that exercise whether or not it is assigned. My advise to teachers is that if you have a class with mostly engineering and science majors, and only a couple of math majors, make a point of isolating the math majors and giving them different homework than the rest, because it is going to be much more important for them in the long run once they get to the higher courses that they have a firm understanding of the processes of proof.

Izzy: I guess we are coming full circle and I was thinking that maybe we can, as people who are already math majors and mathematicians, encourage other freshman and sophomores to keep going into the junior and senior mathematics and to let them know that even after hours of frustration, when you finally get a proof, how incredibly self-satisfying it is. Battling the problem is just a discipline. You can get a great deal of self-satisfaction out of that. If we could just show our peers and classmates that making it through is really satisfying, that would really make a difference. ■


Photograph by Richard Neal

Dan Isaksen, Darren Glass, Izzy Kyle, Jon Scott, and Bob Eslinger

Mathematician at Work

Mathematics can be an end in itself for some people, or it can open a multitude of other doors. The first part of my professional career did not rely heavily on a math background. I flew armed helicopters in Vietnam and was a flight instructor in primary helicopters. But soon the dividends were realized. I was sent to the Army's Guided Missile Systems Officer Course. The heart and soul of the course was applied mathematics, and I was well prepared. Laplace transforms became tools for stability and control analysis, not mathematical abstractions. Calculus of Variations became a means for computing optimum flight profiles. In short, the mathematics took on a concrete reality.

A Masters Degree in Aeronautical Engineering from the University of Texas at Arlington set the stage for my next step, the US Naval Test Pilot School (USNTPS). USNTPS is an ideal mix of the theoretical and the practical. The fine nuances of the stability equations explored in the classroom in the morning, are graphically demonstrated in the air that afternoon, thus cementing the relationship between the mathematics and the real world.

As an experimental test pilot assigned to Edwards Air Force Base, CA, I was fortunate enough to test five prototype helicopters in four years. The highlight was being the project officer and chief test pilot on the Apache attack helicopter. The ultimate test of a mathematical analysis is to bet your life on it by flying an aircraft that isn't quite "house broken" yet.

Following a very stiff competition, I was selected as a NASA Astronaut. My background led to assignment as the Astronaut Office representative to develop the space shuttle Entry Flight Control System, a task I pursued for three years. I found that it was harder to bet someone else's life on a mathematical analysis than it was to bet my own.

The ultimate test of a mathematical analysis is to bet your life on it by flying an aircraft that isn't quite "house broken" yet.

While at NASA, I flew two space shuttle flights. On STS-41B Bruce McCandless and I conducted the first orbital flight tests of the Manned Maneuvering Unit (MMU), the first untethered extravehicular activity from a spacecraft in flight. Being all alone, 1,000,000 feet above the earth, traveling at nearly 17,500 m.p.h., makes one very happy that Isaac Newton and Johannes Kepler were steadfast in their pursuit of mathematics.


Robert L. Stewart

In 1987, I was promoted to Brigadier General. Retired from the Army in 1992, I am now Director of Advanced Programs, Nichols Research Corporation, Colorado Springs.

It should be evident that each step in my career has rested on a firm foundation in mathematics. For me, the study of mathematics was the key that opened the doors to the universe. ■

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Problem Section

Editor

Murray Klamkin
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This section features problems for students at the undergraduate and (challenging) high school levels. As new editor, the problems will of course reflect my tastes as well as the submissions of its readers. My preferences are for problems that are not highly technical so they can be easily understood by the general reader. There should be a certain elegance about the problems; the best problems are elegant in statement ("short and sweet"), elegant in result, and elegant in solution. Such problems are not easy to come by. Nevertheless, any problem submitted should include a solution and be elegant in at least one of the three categories. Original problems are preferred but this does not rule out elegant problems which are not well known (these will be indicated by a dagger (†)). For the latter, any known information about them should be included. Also to be included are "Mathematical Quickies." These are problems which can be solved laboriously but with proper insight and knowledge can be solved easily. These problems will not be identified as such except for their solutions appearing at the end of the section (so no solutions should be submitted for these problems).

All problems and/or solutions should be submitted in duplicate in easily legible form (preferably printed) on separate sheets containing the contributor's name, mailing address, school affiliation, and academic status (i.e., high school student, undergraduate, teacher, etc.) and sent to the editor, Math. Dept., University of Alberta, Edmonton, Alberta T6G 2G1, Canada. If an acknowledgement is desired an e-mail address or a stamped self-addressed postcard should be included (no stamp necessary for outside Canada and the US).

Proposals

To be considered for publication, solutions to the following problems should be received by September 20, 1996.

Problem 46. Proposed by Vaclav Konecny, Ferris State University. Determine all maxima and minima of

$$F(\theta) = \left[1 + p^2 - 2p\cos(2\pi/3 + \theta)\right] \times \left[1 + p^2 - 2p\cos(2\pi/3 - \theta)\right]\left[1 + p^2 - 2p\cos\theta\right]$$

for given $p > 0$.

Problem† 47. Determine the equation of a cone whose vertex is at the center of a given ellipsoid and which passes through all the points common to the ellipsoid and a given concentric sphere.

Problem 48. Proposed by the Problem Editor. Determine all triples of positive integers (a, b, c) such that they are the pth, qth, and rth terms, respectively, of both an arithmetic progression and a geometric progression.

Problem 49. Proposed by P. K. Wagner, Chicago, IL. Determine the extreme values of

$$\frac{A^2a^4\ell^2 + B^2b^4m^2 + C^2c^4n^2}{a^2\ell^2 + b^2m^2 + c^2n^2}$$

where a, b, c are given numbers with $a \geq b \geq c$, $\ell^2 + m^2 + n^2 = 1$, and

$$A = 2\pi abc\int_0^\infty \left\{(a^2+u)^3(b^2+u)(c^2+u)\right\}^{-1/2}du,$$

$$B = 2\pi abc\int_0^\infty \left\{(a^2+u)(b^2+u)^3(c^2+u)\right\}^{-1/2}du,$$

$$C = 2\pi abc\int_0^\infty \left\{(a^2+u)(b^2+u)(c^2+u)^3\right\}^{-1/2}du.$$

Problem 50. Proposed by K. S. Murray, Brooklyn, NY. Ellipses and hyperbolas have the property that the segment of a tangent to either curve intercepted between the two tangents to the curve at ends of its major axis subtends a right angle at a focus. Are there any other smooth curves with this property?

Solutions

Problem 36: An Inequality

Prove that for $x, y \geq 0$,

$$\sqrt[3]{x+\sqrt[3]{x}}+\sqrt[3]{y+\sqrt[3]{y}} \leq \sqrt[3]{x+\sqrt[3]{y}}+\sqrt[3]{y+\sqrt[3]{x}}$$

Solution by the proposer. Although the result follows by showing that an appropriate function has a nonnegative derivative (as done by most of the solvers), a more general result follows easily from the majorization inequality [A. W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and Its Applications,* Academic Press, NY, 1979]. If

$$\begin{gathered} a_1 \geq a_2 \geq \cdots \geq a_n; \\ b_1 \geq b_2 \geq \cdots \geq b_n; \\ \sum_{i=1}^{n-1} a_i \geq \sum_{i=1}^{n-1} b_i, \quad i=1,2,\ldots,n-1; \\ \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i; \end{gathered}$$

then $\sum_{i=1}^n F(a_i) \geq \sum_{i=1}^n F(b_i)$ where F is a convex function. As a special case,

$$\begin{aligned} &F(G(x)+H(x))+F(G(y)+H(y)) \\ &\qquad \geq F(G(x)+H(y))+F(G(y)+H(x)) \end{aligned}$$

where $G(x)$ and $H(x)$ are nondecreasing functions. For the given problem, just let $-F(t)=\sqrt[3]{t}=H(t)$, $G(t)=t$.

Also solved by Steve Benton and Z. L. Zu, T. S. Bolis, Farrell Brumley, Hang Chen, R. J. Covill, David Doster, R. E. Kennedy and Curtis Cooper, D. E. Manes, C. A. Minh, W. C. Reif, and the Western Maryland College Problem Seminar.

Problem 37: A Locus

F is a point and ℓ is a line such that $[F\ell]=3$, where $[F\ell]$ denotes the distance from F to ℓ. Find the locus of the point P such that $PF+[P\ell]=5$.

Solution by David Doster, Wallingford, CT. Let ℓ' be the line through F perpendicular to ℓ and O be the intersection of ℓ and ℓ'. We now choose a rectangular coordinate system such that ℓ is the x-axis and ℓ' is the y-axis. Then $F=(0,3)$. If $P(x,y)$ by any point on the locus, the condition $PF+[P\ell]=5$ is equivalent to $\sqrt{x^2+(y-3)^2}+|y|=5$ or $x^2=16+6y-10|y|$ which in turn is equivalent to $x^2=-4(y-4)$ if $y \geq 0$ and $x^2=16(y+1)$ if $y<0$. Thus any point of the locus lies on parts of two parabolas. Conversely, any point on the latter parts of the two parabolas satisfy the given condition so it is the locus.

Also solved by Benjamin Armbruster (7th grader), Steve Benton and Z. L. Zu, T. S. Bolis, D. K. Johnson, C. A. Minh, Peter Rothmaler (2 solutions), and the proposer. All these solutions were analytic except for the ones by Rothmaler and the proposer who used the definition of a parabola as the locus of the point whose distance from a fixed point equals its distance to a fixed line.

Problem 38: A Smallest Perfect Square

Find the smallest positive integer n such that $19n+1$ and $95n+1$ are both integer squares.

Solution. Eliminating n from $y^2=19n+1$ and $x^2=95n+1$, we get the Pell equation $x^2-5y^2=-4$. Only three solvers, T. S. Bolis (University of Ioannina, Greece), D. K. Johnson (Valley Catholic High School) and the proposer, gave a general solution to the latter. The simplest was the proposer's, i.e., $(x,y)=(x_n,y_n)$ where $\frac{1}{2}(x_n+y_n\sqrt{5})=\{(1+\sqrt{5})/2\}^n$ for $n=1,3,5,\ldots$. It follows that $y_n=F_{2n-1}$ which are the odd terms of the Fibonacci sequence. It then suffices to find the first term of the sequence $2,5,13,34,\ldots$ whose square is congruent with 1 (mod 19). This term is $F_{17}=1597$, so that $n_{\min}=(F_{17}^2-1)/19=134{,}232$.

Editorial note. None of these solvers proved that they had *all* the solutions to the Pell equation; Bolis just referred to the book, H. Hasse, *Number Theory,* Springer-Verlag, pp. 554–555.

The other solvers, Steve Benton and Z. L. Zu, Jeffrey Bonfiglio, D. J. Covert, R. J. Covill, Cadet Freddie Ledfors, D. E. Manes, J. Suck, and the Western Maryland College Problem Seminar, all set up computer programs to give the smallest solution.

Problem 39: Concurrent Lines

Let $n+1$ points be given on a sphere. From the centroid of any n of these points a line is drawn normal to the tangent plane to the sphere at the remaining point. Prove that all these $n+1$ lines are concurrent.

Generalization by T. S. Bolis (University of Ioannina, Greece) and the Problem Editor. An exercise in a geometry book which we do not recall was to prove that if the altitudes of a tetrahedron were concurrent then the perpendicular lines to the four faces at their respective centroids were also concurrent. Another related problem appeared on a Cambridge Scholarship Examination, i.e., if lines are drawn through the midpoints of sides BC, CA, AB of a triangle parallel to PA, PB, PC, respectively, where P is any point in the plane of the triangle, then these three lines are concurrent. More generally, we show that if $P, A_0, A_1, \ldots, A_n$ are $n+2$ points in E^m, and G_i is the centroid of $A_0, A_1, \ldots, A_n$ less A_i, then the lines through G_i parallel to PA_i are concurrent and, conversely, the lines through A_i parallel to PG_i are concurrent ($i = 0, 1, \ldots, n$). A proof follows easily using vectors.

Let $\mathbf{V}$ denote a vector from P to any point V. Then the vector equations of the lines through G_i parallel to PA_i are given by $\mathbf{R}_i = \mathbf{G}_i + \lambda_i \mathbf{A}_i$ where λ_i are real parameters, $\mathbf{G}_i = (\mathbf{S} - \mathbf{A}_i)/n$ and $\mathbf{S} = \mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_n$. By choosing $\lambda_i = 1/n$, it follows that all the lines $\mathbf{R}_i$ have the common point $\mathbf{S}/n$ which lies on the line joining P to the centroid G of all the points $A_0, A_1, \ldots, A_n$. For the converse, the equations of the lines through A_i and parallel to PG_i are given by $\mathbf{R}_i = \mathbf{A}_i + \lambda_i \mathbf{G}_i$. By choosing $\lambda_i = n$, the lines here have the common point $\mathbf{S}$ which again is on the line joining P to G.

It is to be noted that for $n = 2$, the triangle case, we have the following special results: (i) The concurrency of the altitudes and the concurrency of the perpendicular bisectors of the sides each imply the other; (ii) If P is the circumcenter, then the lines through G_i and parallel to PA_i are concurrent at the point $(\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3)/2$ which corresponds to the center of the ninepoint circle of the triangle.

Also solved by D. K. Johnson and the Western Maryland College Problem Seminar.

Problem 40: Hyperbolic Inequality

Prove that

$$\cosh \sqrt{x^2 + y^2} \leq (\cosh x)(\cosh y)$$

for all real x, y.

Solution 1. The solutions by D. K. Johnson (Valley Catholic High School), D. E. Manes (SUNY at Oneonta), and E. T. H. Wang (Wilfrid Laurier University) were the same and used the power series for $\cosh x$, i.e.,

$$\begin{aligned}
\cosh \sqrt{x^2 + y^2} &= \sum_{n=0}^{\infty} (x^2 + y^2)^n/(2n)! \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{2k} y^{2n-2k}/(2n)!, \\
(\cosh x)(\cosh y) &= \left\{ \sum_{n=0}^{\infty} x^{2n}/(2n)! \right\} \left\{ \sum_{n=0}^{\infty} y^{2n}/(2n)! \right\} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} x^{2k} y^{2n-2k}/(2k)!(2n-2k)!
\end{aligned}$$

It then suffices to show that

$$\binom{n}{k} \leq \binom{2n}{2k}$$

for any positive integer n and $0 \leq k \leq n$ and this is done by induction.

Solution 2 by T. S. Bolis (University of Ioannina, Greece). Since $\cosh \sqrt{t}$ is convex for $t \geq 0$, it follows by Jensen's inequality that $\cosh \sqrt{u} + \cosh \sqrt{v} \geq 2 \cosh \sqrt{(u+v)/2}$. Setting $\sqrt{u} = x + y$ and $\sqrt{v} = x - y$ and expanding out, we obtain the desired inequality for $x \geq y \geq 0$. Then by symmetry and cosh being an even function, we get the general case.

Solution 3 and comments by the proposers. Holding x fixed, consider the function of y:

$$F(y) = (\cosh x)(\cosh y)/\cosh \sqrt{x^2 + y^2}.$$

Since $F(0) = 1$ and $F(\infty) = \cosh x \geq 1$, it suffices to show that F is increasing in y for $y \geq 0$, or equivalently that $\partial F/\partial y \geq 0$. The latter condition reduces to $(\tanh y)/y \geq (\tanh \sqrt{x^2 + y^2})/\sqrt{x^2 + y^2}$ and this follows since $(\tanh y)/y$ is a decreasing function of y (just consider its derivative).

By letting $x = u^2 - v^2$ and $y = 2uv$, one can show that another equivalent form of the inequality is $(\tanh u^2)(\tanh v^2) \leq \tanh^2 uv$.

The geometric significance of the given inequality is that if a, b, c are the sides of a hyperbolic right triangle with hypotenuse c, then $c^2 \geq a^2 + b^2$ and with equality only for a degenerate triangle. Note that here $\cosh c = (\cosh a)(\cosh b)$. Analogously, for spherical triangles (elliptic geometry) we have $c^2 \leq a^2 + b^2$ and since here $\cos c = (\cos a)(\cos b)$, this leads to $\cos \sqrt{a^2 + b^2} \leq (\cos a)(\cos b)$ subject to $a^2 + b^2 \leq \pi^2$ and vice-versa. Also $(\tan x^2)(\tan y^2) \geq \tan^2 xy$ subject to $x^2, y^2 \leq \pi/2$.

Also solved by Western Maryland College Problem Seminar.

Problem† 47: (Quickie) Equation of a Cone

If the equations of the ellipsoid and sphere are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = r^2,$$

the equation of the cone is

$$x^2 \left(\frac{1}{a^2} - \frac{1}{r^2} \right) + y^2 \left(\frac{1}{b^2} - \frac{1}{r^2} \right) + z^2 \left(\frac{1}{c^2} - \frac{1}{r^2} \right) = 0.$$

Note that since the equation is homogeneous of degree 2, it represents a cone whose vertex is the origin; and it is clear that the cone intersects the sphere and the ellipsoid in the same points.

If say $a > b > c$ and then $r = b$, the cone is degenerate and consists of two intersecting planes which intersect the ellipsoid in a pair of circles of radius b and which are the largest circles lying on the ellipsoid. This is related to Putnam Competition Problem A-5, 1970 which was to determine the radius of the largest circle on the ellipsoid.