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math HORIZONS

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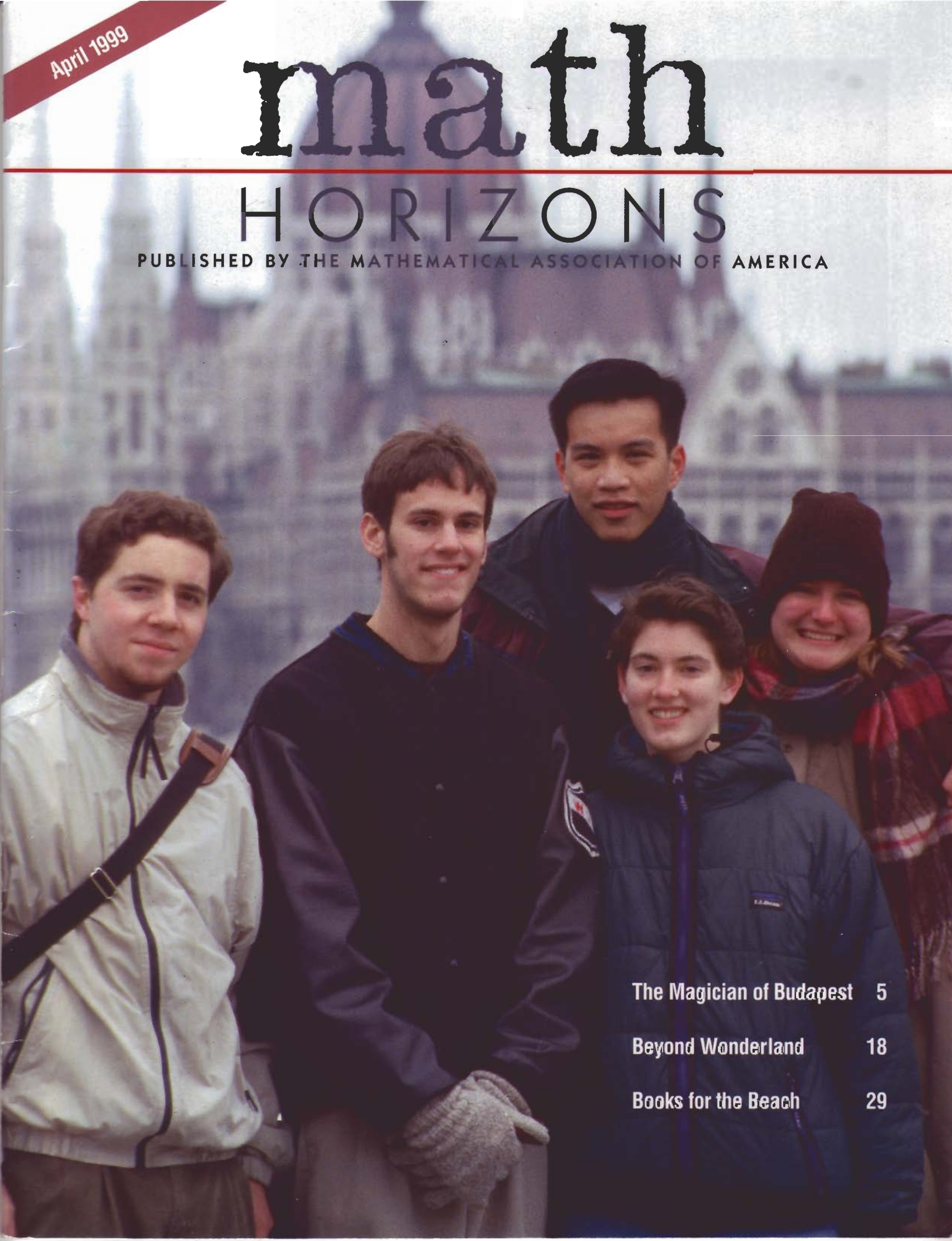

Math Horizons is for undergraduates and others who are interested in mathematics. Its purpose is to expand both the career and intellectual horizons of students.

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In this issue

On the cover: Students from the Budapest Semesters in Mathematics Program in front of Parliament. Photo by Kawa Kamal.

math
HORIZONS

Instructions for Authors

Math Horizons is intended primarily for undergraduates interested in mathematics. Our purpose is to introduce students to the world of mathematics outside the classroom. Thus, while we especially value and desire to publish high quality exposition of beautiful mathematics we also wish to publish lively articles about the culture of mathematics. We interpret this quite broadly—we welcome stories of mathematical people, the history of an idea or circle of ideas, applications, fiction, folklore, traditions, institutions, humor, puzzles, games, book reviews, student math club activities, and career opportunities and advice. We welcome electronic submission of manuscripts. Please send to skennedy@carleton.edu. If submitting by mail, please send two copies to Steve Kennedy, Department of Mathematics and Computer Science, Carleton College, Northfield, MN 55057.

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PETER SCHUMER

The Magician of Budapest

As the millennium draws to a close, many people are busy creating and debating "top ten" lists — greatest movies of all time, best fiction books of the decade, most successful musicians of the century, most significant inventions, and so on. One name which would appear on almost all such rankings of the top ten mathematicians of this century would be that of Paul Erdős (pronounced, approximately, Air Dish). In his lifetime, Erdős wrote or co-wrote nearly 1500 mathematical articles (the equivalent of a research paper every two weeks for 60 years!) He did significant work in number theory, geometry, graph theory, combinatorics, Ramsey theory, set theory, and function theory. He helped create probabilistic number theory, extremal graph theory, the probabilistic method, and much of what is now referred to as discrete mathematics. His influence on fellow mathematicians and on mathematics as a whole is bound to last for centuries to come. His reputation for being published in so many journals and in so many languages led to the following limerick:

A conjecture thought to be sound
Was that every circle was round
In a paper of Erdős
Written in Kurdish
A counterexample is found!

Epsilon Years

Paul Erdős was born in Budapest, Hungary on March 26, 1913, the son of math and physics teachers, Anna and Lajos Erdős. Paul's two older sisters died of scarlet fever while his mother remained in the hospital following his birth. This family tragedy resulted in a very close but over-protective home environment where Paul was home-schooled and his mathematical genius was able to flourish. Unfortunately, it also resulted in a socially awkward and eccentric individual who depended heavily on the care and goodwill of his many friends sprinkled across the globe.

Like the great German mathematician Carl Friedrich Gauss, Erdős's mathematical talents blossomed early. At age three Paul discovered negative numbers when he correctly subtracted 250 from 100. By age four he could multiply three-digit numbers in his head. He often entertained family friends by asking them for their birthday and then telling them how many seconds they had been alive.

At a young age Paul's father taught him two theorems about primes (i) that there are infinitely many primes, while at the same time (ii) there are arbitrarily large gaps between successive primes. To Paul, the results seemed almost paradoxical, but they led to a deep fascination with prime numbers and to a quest for a better understanding of their complicated arrangement. Again, like Gauss, an early fascination with number theory was the impetus for Erdős's lifetime dedication to the world of mathematics.

Another early mathematical influence was the *Hungarian Mathematical Journal for Secondary Schools*, commonly referred to as *KoMal*. The most popular part of the journal was a regular problem section where student solutions were pub-

Paul Erdős delivering his "Sixty Years of Mathematics" lecture at Trinity College, University of Cambridge in June, 1991, the day before receiving Cambridge's prestigious honorary doctorate. Photo by George Csicsery from his documentary film N is a Number: A Portrait of Paul Erdős.

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lished and name credit given for correct solutions. At the end of the year the pictures of the most prolific problem solvers were included in the journal. In this way, the best mathematics and science students in Hungary were introduced to one another. In this sense, Paul's "publications" began when he was barely thirteen years old.

The Magician of Budapest

Paul's first significant result was a new elementary proof of Bertrand's Postulate, which he discovered as an 18-year-old university student. The French mathematician, J.L.F. Bertrand, a child prodigy himself, conjectured that for any natural number $n > 1$, there was always a prime between n and $2n$. Bertrand verified the result up to $n = 3{,}000{,}000$, but was unable to prove it. Five years later, in 1850, the Russian P.L. Chebyshev proved what has become known as Bertrand's Postulate. The proof was difficult and relied heavily on analytic methods (function theory and advanced calculus). Erdős created a new proof which, though quite intricate, was elementary in the sense that no calculus or other seemingly superfluous analytical methods were used. This proof combined with other related results on primes in various arithmetic progressions constituted his doctoral dissertation.

Another early success was a generalization of an interesting observation by one of his friends. Esther Klein noticed and proved that for any five points, no three collinear, in the plane, it is always the case that four of them can be chosen which form the vertices of a convex quadrilateral. What Erdős and George Szekeres were able to show was that for any n there is a number N, depending only on n, so that any N points in the plane (with no three collinear) have a subset of n points forming a convex n-gon. Since Esther and George became romantically involved during this period and later married, the result was dubbed the Happy End Problem. Furthermore, Erdős and Szekeres conjectured that in fact the smallest such N will always be $N = 2^{n-2}+1$. Interestingly, the more general conjecture still has not been proven. However, the Happy End Problem was a harbinger of much of Erdős's later work — fruitful collaborations and fascinating conjectures. A good theorem often creates more questions than it answers.

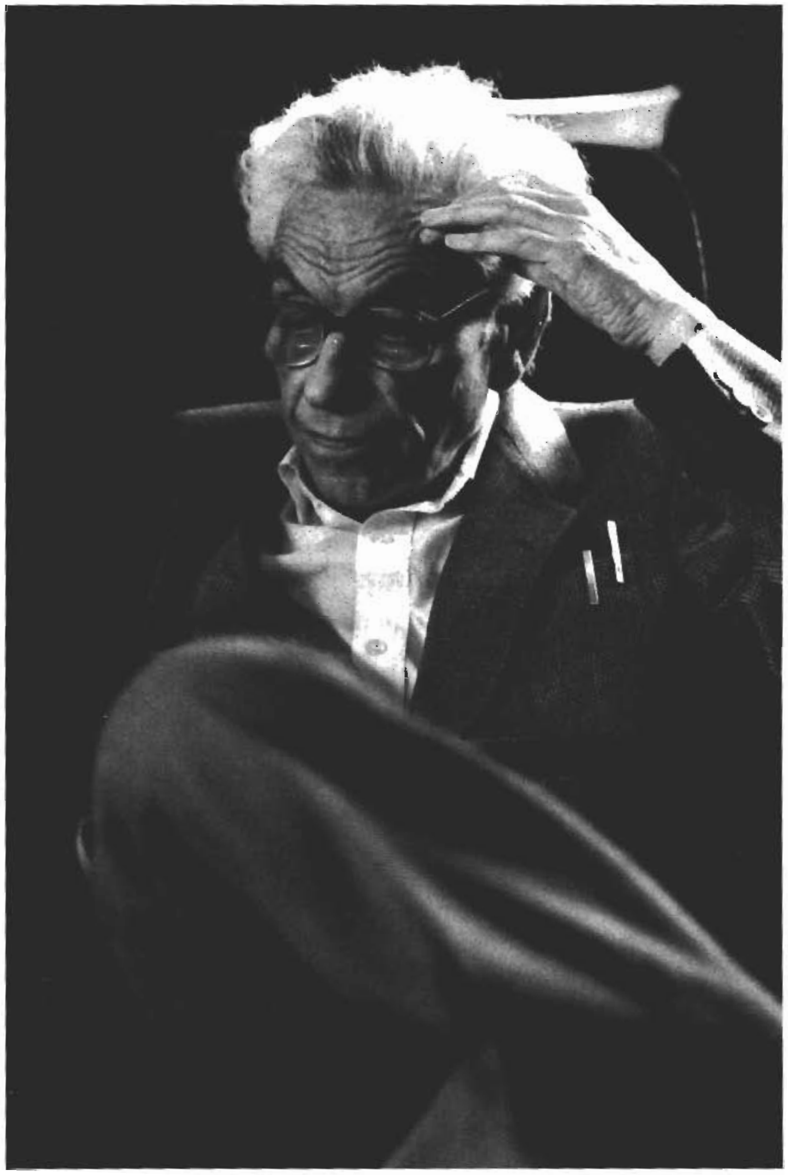


Photo by George Csicsery from his documentary film N is a Number: A Portrait of Paul Erdős.

Erdős also proved a neat result about abundant numbers. Let $s(n)$ represent the sum of the proper divisors of n (i.e., all positive divisors except n itself). Then n is called *deficient, perfect,* or *abundant* if $s(n)$ is less than, equal to, or greater than n, respectively. Such numbers have been studied since the time of Pythagoras. The German mathematician Issai Schur conjectured that the set of abundant numbers had positive density. That is, let $A(x)$ be the number of abundant numbers less than or equal to x, then Schur's conjecture states that $\lim_{x\to\infty} A(x)/x$ exists and is strictly greater than 0. Erdős's brilliant proof of this result led Schur to dub Erdős "the Magician of Budapest."

With life in Hungary ever worsening for Jewish intellectuals, Erdős obtained a fellowship to the University of Manchester in England. In 1934 Erdős traveled there via Cambridge University and began his mathematical travels and worldwide collaborations which never let up until his death. It was fourteen years before Erdős was able to return to Budapest where his mother had miraculously survived the war. Unfortunately, Paul's father died during that period and four out of five of his aunts and uncles perished in the Holocaust. Paul's beloved mother spent much of the rest of her life traveling with her son and using her apartment as a repository for his ever-increasing mountain of reprints.

Though Erdős's relationship with the American government was generally harmonious, it wasn't so during the McCarthy era. In 1954 while on a temporary faculty position at Notre Dame, Erdős wished to attend the International Congress of Mathematicians in Amsterdam. Knowing he came from a Communist country, an agent from the Immigration and Naturalization Service interviewed Erdős and asked him what he thought of Karl Marx. Erdős replied, "I'm not competent to judge. But no doubt he was a great man." Perhaps due to this, Erdős was denied a re-entry visa after attending the Mathematical Congress. Strong support and letters to state senators from many American mathematicians finally resulted in Erdős being allowed to return to the U.S. in 1959. From that point on, he could come and go freely.

His Brain is Open

To all who knew Erdős, it appeared that he spent ninety-nine percent of his wakeful hours obsessed with mathematics (though he somehow developed a deceptive skill at both table tennis and the game of Go). Twenty hours of work a day was not at all unusual. Upon arriving at a meeting, he would announce, in his thick Hungarian accent, "my brain is open." At parties, he would often stand alone oblivious to all else, deep in thought pondering some difficult argument. When being introduced to a math graduate student, it was not unusual for him to ask, "What's your problem?" One would normally be taken aback by such a remark if it were uttered by a stranger in less than friendly surroundings, but with Erdős it was clearly meant as a friendly "hello." He was taking you seriously as a fellow dweller in the world of mathematics.

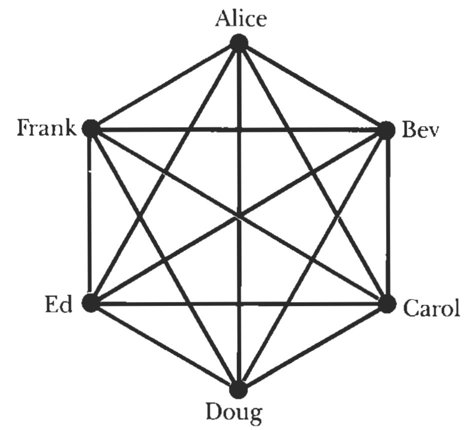
One of Erdős's greatest triumphs was his elementary proof of the Prime Number Theorem (PNT). The PNT describes the asymptotic distribution of the prime numbers and variants of it were conjectured by both Gauss and Legendre in the late 1700's. Specifically, let $\pi(x)$ be the number of primes less than or equal to x and let $Li(x) = \int_2^x dt/\log t$. The PNT states that $\lim_{x\to\infty} \pi(x)/Li(x) = 1$, that is, the number of primes less than x is asymptotic to $Li(x)$.

Significant progress towards a proof of the PNT was made by Chebyshev in the 1850's and by G.B. Riemann in 1859. Riemann's contribution was based on a deep and careful study of the complex-valued zeta function. Finally, in 1896 the French mathematician J. Hadamard and the Belgian mathematician C.J. de la Vallée Poussin each proved the result using delicate arguments from complex function theory. Many feel that the proof of the PNT was the mathematical capstone of the nineteenth century.

In the first half of the twentieth century, the search for an "elementary" proof of the PNT seemed hopeless. However, in 1949 Paul Erdős and the Norwegian mathematician Atle Selberg, working in tandem but not together, found such proofs. Selberg won a Fields Medal for his work while Erdős won the prestigious Cole Prize in algebra and number theory for his contribution.

Another area which fascinated Erdős was classical Ramsey theory which describes the number of ways of partitioning a given set into a given number of subsets. The party problem is the classic example: at a party with six individuals, prove that there must always be at least three mutual friends or three mutual strangers. More generally define $r(u,v)$ to be the smallest integer r such that if the edges of a complete graph on r vertices are colored one of two colors, then there must be a complete subgraph on u vertices of one color or a complete subgraph on v vertices of the other color. Thus, the solution to the party problem amounts to proving that $r(3,3) = 6$. The values for $r(3,n)$ are also known for $n = 4$, 5, 6, 7, and 9. In addition, it has been shown with significantly more work that $r(4,4) = 18$. Erdős showed $r(k,l) \leq C(k+l-2, k-l)$, the number of combinations of $k+l-2$ objects chosen $k-l$ at a time. Erdős offered \$250 to anyone who could prove that $\lim_{n\to\infty} r(n,n)^{1/n}$ exists. If the limit does exist, it's known to be between $\sqrt{2}$ and 4. Interestingly, the value of $r(5,5)$ is still unknown, though it must be at least 43 and at most 49. Erdős was fond of saying that if an evil spirit was going to destroy the world unless we could determine $r(5,5)$, then it would be prudent for all of humanity to devote all of its resources to this problem. On the other hand, if the evil spirit insisted on knowing $r(6,6)$, then it would be best for all of us to try to destroy the evil spirit!

Erdős not only loved working on difficult problems and proving theorems, but always strived for the most elegant and direct proof. He had unique religious views and referred to the Almighty as the



The party problem: if blue edges connect friends and black edges connect strangers, then Alice, Bev, and Ed are mutual strangers. There is no way to color the edges without forming either a blue or black triangle.

Mathematicians are often divided into two camps: theory builders and problem solvers. Erdős was definitely a problem solver. Here are a few of Erdős's many theorems.

1. There are infinitely many odd integers that are not of the form $p + 2^k$ where p is a prime.
2. The product of two or more consecutive positive integers is never a square or any other higher power.
3. A connected graph with minimum degree d and at least $2d+1$ vertices has a path of length at least $2d+1$.
4. Let p_n be the nth prime number. Then the set of limit points of the set $\{(p_{n+1} - p_n)/\log n\}$ has positive density.
5. Let $d(n)$ be the number of positive divisors of n. Then $\sum_{n=1}^{\infty} d(n)/2^n$ is irrational.

SF (or *Supreme Fascist*). Erdős felt that he was forever in the midst of an ongoing personal battle with the SF. However, one positive aspect was that the SF kept a secret book, *The Book*, which had all the theorems that would ever or could ever be discovered along with the simplest and most elegant proofs for each one. The highest compliment Erdős would give someone was that their proof was "one from The Book."

Nothing was more exciting to Erdős than to discover a mathematically talented child and to excite him or her about mathematics. In 1959 Erdős arranged to have lunch with a very precocious 11-year old, Lajos Posá. Erdős challenged the youngster to show why if $n+1$ integers are chosen from the set $\{1, 2, ..., 2n\}$, then there must be two chosen numbers which are relatively prime. Clearly the set of even numbers less than or equal to $2n$ does not have this property and shows that just choosing n such numbers is not sufficient. According to Erdős, within half a minute Posá solved the problem by making the striking observation that two consecutive integers must always be chosen. Erdős commented that rather than eating soup, perhaps champagne would have been more appropriate for this occasion!

Erdős's greatest influence on fellow mathematicians, young and old alike, was his continual outpouring of new conjectures coupled with various monetary rewards. Some problems had a price tag of just a few dollars while others went for several thousand dollars. To solve an Erdős problem is considered a great accomplishment, and the larger the reward the more difficult Erdős considered the problem to be. Erdős was often asked what would happen if all his problems were solved simultaneously. Could he possibly pay up? His answer was, "Of course not. But what would happen if all the depositors went to every bank and demanded all their money. Of course the banks could not pay — and besides, it's more likely that everyone will simultaneously ask for their money than that all of my problems will be solved suddenly."

Here are a few outstanding conjectures and open problems that Erdős left us:

1. Do there exist infinitely many primes p such that every even number less than or equal to $p-3$ can be expressed as the difference between two primes each at most p? For example, 13 is such a prime since $10 = 13-3$, $8 = 11-3$, $6 = 11-5$, $4 = 7-3$, and $2 = 5-3$. The smallest prime not satisfying this condition is $p = 97$.
2. For every integer n are there n distinct integers for which the sum of any pair is a square? For example, for $n = 5$, the numbers: -4878, 4978, 6903, 12978, and 31122 have this property.
3. Can you find a polynomial $P(x)$ for which all sums $P(a) + P(b)$ are distinct with $0 \leq a < b$? For example, $P(x) = x^3$ doesn't work since $10^3 + 9^3 = 12^3 + 1^3$. However, $P(x) = x^5$ is considered a likely candidate.
4. Are there infinitely many primes p for which $p - n!$ is composite for all n such that $1 \leq n! < p$? For example, when $p = 101$, $101 - n!$ is composite for $n = 1, 2, 3$, and 4.
5. A natural number n is *pseudoperfect* if it is abundant and it is also the sum of some of its proper divisors. For example, $n = 66$ is pseudoperfect since $66 = 11 + 22 + 33$. A number is *weird* if it is abundant but not pseudoperfect. Try verifying that $n = 70$ is a weird number (sorry Mark McGwire). For ten dollars, are there any odd weird numbers?
6. A system of congruences $a_i \pmod{n_i}$ where $n_1 < n_2 < \cdots < n_k$ is a *covering system* if every integer satisfies at least one of the congruences. For example, 0 (mod 2), 0 (mod 3), 1 (mod 4), 5 (mod 6), and 7 (mod 12) form a covering system. Erdős conjectured that for every c, there is a covering system with $n_1 \geq c$. (Currently $c = 24$ is the largest value that has been constructed.) An open question of Erdős and J. Selfridge asks whether there is a covering system with all moduli odd.
7. A \$5000 conjecture: Let $A = \{a_i\}$ be any sequence of natural numbers for which $\sum_{i=1}^{\infty} 1/a_i$ diverges. Is it true that A must contain arbitrarily long arithmetic progressions? If so, one consequence would be that the set of primes contained arbitrarily long arithmetic progressions. Currently the record is 22 consecutive primes.

Poor Great Old Man

Erdős was almost as well-known for his eccentricities as he was for his brilliant mind. By his own admission, Erdős never attempted to butter his own toast until he was already an adult. "It turned out not to be too difficult," he admitted. Many mathematicians have had the experience of either walking or driving Erdős to his next commitment, only to learn after some time that he assumed you knew where he was supposed to be going.

Erdős had an aversion to old age and infirmities and an obsession with death. When breaking off work for the night he would say, "We'll continue tomorrow — if I live." At age 60, Erdős started appending acronyms to his name. The letters *pgom* stood for "poor great old man." Five years later he added *ld* for "living dead" and so on. Eventually he got to *cd* denoting "counts dead." Erdős explained this as follows: The Hungarian Academy of Sciences has a strict limit on the total number of members that it can have at one time. However, once you reach the age of 75, while you are still a member, you are not counted against the total. Therefore, at that point, you're counted as if you were dead.

When asked "how old are you?" Erdős would answer that he was two-and-a-half billion years old. After all, when he was a child he was taught that the earth was two billion years old and now they say it's four-and-a-half billion years old! Some say that Erdős was the Bob Hope of mathematicians. Not only did he share humorous stories, but he spent a great deal of time traveling and, by lecturing, which he called *preaching*, raising the morale of the mathematical troops.

Erdős was well-known for a shorthand language many call Erdősese. An *epsilon* was a child, *poison* meant alcoholic drink (which he carefully avoided), *noise* meant music, *boss* was wife, and *slave* husband. If someone were *captured* that meant they got married, while *liberated* stood for divorced. If a mathematician stopped publishing he *died*, while actually dying was referred to as *having left*. Nothing bothered Erdős more than political strictures which did not allow for complete freedom of expression and the ability to travel freely. Erdős referred to the Soviet Union as *Joe* (for Joseph Stalin) and the United States as *Sam*.

No account of Paul Erdős would be complete without mentioning the concept of Erdős numbers. Erdős himself had Erdős number zero. Anyone who co-authored a paper with him (there are at least 472 such people) have Erdős number one. Those who do not have Erdős number one, but co-authored a paper with someone who does, are assigned Erdős number two (there are at least 5000 such people), and so on. The largest Erdős number believed to exist is seven. Erdős himself added an interesting wrinkle for those having Erdős number one. He claimed that one should instead be assigned Erdős number $1/n$ if you co-authored n papers with him. The lowest Erdős number in this case would be held by Andras Sarkozy with number 1/57 — just edging out Andras Hajnal with number 1/54. (For more on Erdős numbers, see [3].)

Having left

Paul Erdős received countless honorary degrees and his work was and continues to be the focus of many international conferences. In 1984, Erdős received the prestigious Wolf Prize for his lifetime's contributions to the world of mathematics. Of the $50,000 awarded, he immediately donated $49,280 to an Israeli scholarship named in his mother's honor. On other occasions, he donated money to Srinivasa Ramanujan's widow, to a student who needed money to attend graduate school, to a classical music station, and to several Native American causes. Always traveling with a single shabby suitcase which doubled as a briefcase, he had little need or interest in the material world. He had no home and nearly no possessions. Without hesitation, he once asked the versatile Canadian mathematician Richard Guy for a $100 adding, "You're a rich man." Richard Guy gladly gave him the money. Later Guy poignantly noted, "Yes, I was. I knew Paul Erdős."

Paul Erdős was a very versatile and creative mathematician. The vast quantity of his research output alone qualifies him as the Euler of our age. He far surpassed Einstein's ultimate litmus test for success which was to be highly esteemed by one's colleagues. Once during a lecture by the late number theorist Daniel Shanks, a long computer-generated computation resulted in a 16-digit number. Shanks, who was not known for sprinkling praise lightly said, "I don't know if anyone really understands numbers like these — well, maybe Erdős."

Paul Erdős's fantasy was that he would die in the midst of a lecture. After proving an interesting result, a voice from the audience would pipe up, "but what about the general case?" Erdős would reply, "I leave that to the next generation" and then immediately drop dead. In fact, Erdős left us on September 20, 1996 while attending a conference in Warsaw. Let's hope he is now realizing his dream of collaborating with Euclid and Archimedes simultaneously. They'd probably relish having an Erdős number of one.

Was he one of the great mathematicians of the century? Echoing Erdős himself, I'm not competent to judge. But no doubt he was a great man. ■

For More Information

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PAUL D. HUMKE

Math Majors Study Abroad in Beautiful Budapest

Before the Budapest Semesters in Mathematics Program

Things were different in Budapest in 1983 when Gyuri Petruska and I first discussed the idea of St. Olaf undergraduates studying mathematics in Budapest: no internet, no e-mail, no fax machines (very few phone lines!), a Warsaw Pact, rent and heat at essentially no charge, and a world class opera absolutely anyone could afford. But, some things were not so different: powerful and distinctive intellectual communities, a deeply rooted tradition of ferreting out and developing mathematical talent, colorful and effective institutions maintaining world class standards in the face of forbidding obstacles, a common opinion that soon all would be lost. I had spent several extended stays in Budapest doing research and by 1983, Petruska and I knew each other pretty well, but he really got my attention when out of the blue he asked:

> If there was a mathematics program in Hungary for St. Olaf students, would anyone come?

This question contained the electricity of a real idea! Certainly in 1983, the popular press in the United States was so convinced of U.S. pre-eminence in mathematics and science that excellence in other programs was all but invisible. Moreover, Hungary was *behind the Iron Curtain* and parents would be wary of allowing their children to live there. The fact that Petruska had asked me this particular question meant to me that he and probably his Hungarian colleagues as well were serious about creating an international opportunity for North American undergraduate mathematics students.

At that point, Petruska and I were thinking in terms of a small scale semester exchange program between St. Olaf and Eötvös University, but a larger idea was taking hold.

In the Beginning

When Petruska returned to Budapest he spoke with colleagues about the discussions he'd had with me and they responded by outlining what soon was to become BSM. Immediately, Petruska lent his support to the new project.

A new government regulation had come out in Hungary according to which universities would be allowed to offer fee-based programs to foreign students from the West. A successful program had already been developed for German medical students and the Ministry of Education was encouraging additional programs in several areas including mathematics. Most important, and quite uncharacteristic of a Communist government, the universities would retain a degree of control over how the tuition fees would be spent. Encouraged by this prospect, Vera Sós and Laci Lovász became the engines behind a plan to create a school of mathematics for foreigners in Hungary. But the target audience (or even target country!) was not clear at all. In a letter to Laci Babai, a friend from Eötvös University visiting Oregon that fall, Lovász wrote that he felt Masters or PhD students could be counted on and perhaps a curious undergraduate or two. Sós, on the other hand, was more optimistic about an undergraduate audience. The critical idea came from Gene Luks, Babai's friend and colleague at the University of Oregon with whom he spoke immediately after receiving the letter from Lovász. Luks recognized that the right niche was the "semester abroad" for undergraduates. He argued that this niche was tailor-made for an Hungarian program: it was a concept Americans were familiar with and all that was needed was to fill it. It was **the** solution and, as Babai recalled later, "from that moment I was totally committed to creating this program."

Babai coined the name, "Budapest Semesters in Mathematics," and set out to design the curriculum, the brochure,

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`http://www.stolaf.edu/depts/math/budapest.html`

Szent Mattias Church on Castle Hill in Buda.

the advertising strategy, etc. As he wrote me of this period:

> The curriculum was relatively easy to design; emphasis on combinatorics and number theory came naturally both because they are Hungarian strengths and because they are generally neglected in American curricula. I decided that we should use Paul Erdős's name and fame, he would certainly warmly embrace the endeavour (as he indeed did), so one subject should be called "Conjecture and Proof," adapting Erdős's favorite phrase. I thought the course would be an adaptation of a course taught by Pósa at Eötvös University to math education students; eventually it was Laczkovich who filled the title with wonderful contents. History, arts, and Hungarian language seemed the natural choices of non-mathematical subjects.
>
> The brochure was a critical task. My English was quite limited at the time, this made the task several orders of magnitude more difficult. Gene Luks was immensely helpful. I would write a page into the computer, he would correct every phrase, suggest alternatives, I would rewrite it and add another page, etc. After a dozen iterations which consumed my entire holiday season, the brochure was ready, and with it a detailed outline of the program. It was clear that the target starting date of September '84, requested in Lovász's letter, was impossible to meet, so I set the beginning of the program at February, 1985. My communication with Budapest was limited; letters were too slow and the telephone too expensive, e-mail to Hungary did not exist. I knew I was doing the right thing, those at home had to implement it.

The First Class

Babai's proposed BSM Program found immediate and enthusiastic support back in Budapest and at this point, discussion and planning turned to focused action. It was by no means easy.

Recruiting first-rate faculty to design courses for the future program was not hard. After all, enthusiastic scholar-teachers were the foundation of the entire enterprise. Finding appropriate institutional support was the difficult part. Lovász, Sós, Petruska, and many others worked long and difficult hours trying to identify an appropriate institutional host for the new program. The danger at Eötvös was excessive bureacracy and a rigid power structure; what BSM needed was great flexibility and independence. Several times all seemed lost and then a key person or idea came forward which enabled planning to continue. The Director of the Institute of Postgraduate Studies at the Technical University—Budapest, Atilla Horváth, showed courage, wisdom, and foresight by supporting the program from its inception. Horváth, would eventually provide the program with an ideal institutional host, providing classroom and office space, administrative support, and the great flexibility that has been critical to the program. He is but one of many who took leadership; on several occasions, Paul Erdős gave generously of his time to support this program as yet unborn.

As Babai told me,

> I still had the task of shouting into America's ears, "Here we are, it's great stuff."

The "great stuff" was not clear to the average American, but it **was** generally known to mathematicians.

1. Hungary has a unique mathematical culture which intimately combines excellence in research with excellence in teaching.
2. Budapest, traditionally a bridge between East and West, is a beautiful and fascinating city. Moreover, because of Communism, Budapest was safe.

But from the perspective of recruiting mathematics undergraduates to Hungary, the key, but hardly secret, ingredient was Paul Erdős. Erdős was not merely respected by those who knew him, he was loved. And because he was Erdős, nearly everyone knew him! As Babai put it in the very first BSM literature, students would be studying in *the Country of Paul Erdős*.

Babai needed help and he knew it. Without a penny of funding he had to rely on volunteers and he found them in the collaborators of Erdős, and those who had worked with

Erdős's collaborators. Indeed, most of those active in the early organization of BSM, including Lovász and Sós, fell into one of those two categories.

Babai asked several leading American mathematicians to lend their names to the Program and compiled an impressive North American Advisory Board. Tom Trotter agreed to serve as North American Director and immediately went to work establishing the North American headquarters at the University of South Carolina.

Later, I replaced Tom as *North American Director*, Bonnie Humke became *Program Administrator* and St. Olaf College became the North American headquarters serving as *Agent College* for the Program. Laci Babai continues as the *Program Coordinator*, while Gyuri Petruska now serves as *Hungarian Director*. The Program has continued its cooperation with the Technical University—Budapest, but now through the new College International.

Gradually the program increased to its current size of about 35 to 45 students each semester. At latest count, more than 170 colleges and universities have sent students to the program, many of them several times. These institutions range from regional and national liberal arts colleges like Smith, Ripon, Elon, Hope, Pomona, and Redlands to regional and national universities like Waterloo, Nebraska, Harvard, Washington State, Michigan, and Princeton.

Program Goals of the BSM

The goals of the BSM Program have not changed much over the years, but how we've tried to accomplish them has varied considerably. Indeed, the continuing dynamism of the Hungarian political and social environment has kept us on our toes. What are the Goals for BSM?

- To provide highly motivated North American undergraduate students of mathematics an opportunity to experience the mathematical as well as the general culture of Hungary. To accomplish this it is important to:
  1. Attract the best Hungarian mathematicians to teach in the Program. (Traditionally these mathematicians hold positions at the Mathematical Institute or Eötvös University.)
  2. Attract North American students who are serious about mathematics, likely to benefit from the experience of mathematics Hungarian style, and who are excited about experiencing the cultural adventure of living in Budapest.
- To offer housing and living conditions which will be safe and which will embed the students in the larger Hungarian culture.
- To provide an academic environment concomitant with North American student expectations, including library and internet access.

Brief Program Description

During the past decade, participation in the BSM Program has grown to enroll thirty-five to forty-five students each semester. The mean home school grade point average of a participant is just over 3.7 on a scale of 4, with mathematics grade point slightly higher. Other than the fact that participants are talented and motivated in mathematics, they are the most heterogeneous group I have ever been associated with. In general, BSM students are an adventurous lot who tend to engage their chosen activities with spirited determination and energy. Several recent participants have used their

Reunion of BSM participants in San Antonio (January 1999). Program Director and Program Administrator Paul and Bonnie Humke are on the left.

"spare time" to play in one of the Liszt Academy's orchestras, others have served Hungarian relief organizations aiding refugees from Romania and countries of the former Yugoslavia, several have joined sports teams in swimming, soccer, fencing; all have taken the opportunity to investigate various parts of Hungary and the surrounding European countries. Most of our students attend their first opera in Budapest and many become devotees of the classical musical and intellectual culture which is still vibrant and affordable. Although the BSM Program does not organize outings we do post information concerning cultural events and show students where they can purchase tickets, but little encouragement is needed. These students are competent and engaged!

Two semesters are offered each year; each semester comprises fourteen weeks of teaching and one week of comprehensive examinations. Fall Term begins the first week of September and ends in mid-December, while the Spring Term begins the first week of February and ends in May. There are midterm breaks in each semester. An intensive Hungarian language course, offered by the Babilon School of Languages, begins about two weeks prior to the beginning of each semester. Although this course is optional, students who attend emerge from the non-credited eighty hours with a solid survival Hungarian.

Students receive orientation materials from the North American Office and both an orientation packet and lecture/discussion at the beginning of each term in Budapest.

Students normally take three to four mathematics courses and one or two intercultural courses each semester. The BSM Program offers Beginning and Advanced Hungarian Language, Central European History and a Hungarian Culture course each semester. About twenty additional nonmathematics courses are available to BSM students through other American programs taught at the College International. A complete listing of these can be found at our website.

The mathematics courses offered by BSM vary slightly from semester to semester depending on what preregistration choices the students have selected and also which instructors are in Hungary at the time. In the fall semester of 1998, for example, the courses offered were: Algebra, Analysis, Complex Functions, Combinatorics, Conjecture and Proof, Geometry, Graph Theory, Number Theory, Probability, Real Functions, Set Theory, Statistical Methods, and Theory of Computing. Each course meets 3 hours per week. Classes are taught in English by eminent Hungarian professors, most of whom have had teaching experience in North American universities. In keeping with Hungarian tradition, teachers closely monitor each individual student's progress. Considerable time is devoted to problem solving and encouraging student creativity. Emphasis is on depth of understanding rather than on the quantity of material.

The imprint of the Hungarian tradition is particularly prominent in some of the courses. Combinatorics and Topics in Graph Theory concentrate on combinatorial structures and algorithms, a stronghold of Hungarian mathematics. These courses, along with Theory of Computing, are a valuable introduction to theoretical computer science. Number Theory, especially the advanced course Number Theory B, displays the mark of Paul Erdős's profound influence on the subject.

Conjecture and Proof, even more than other courses, introduces the student to the excitement of mathematical discovery. Concepts, methods, ideas and paradoxes that have startled or puzzled mathematicians for centuries are reinvented and examined under the guidance of enthusiastic and experienced instructors. The topics covered range from ancient problems of geometry and arithmetic to 20th-century measure theory and mathematical logic.

Upon completion of the program, students receive an American-style transcript which lists courses taken and a grade. Normally, official transcripts are also sent directly from the North American Office both to the student and to the student's home institution. Course materials are designed so that credits will be easily transferable to North American colleges and universities.

Broadening Our Horizons in Budapest

Despite our problems, we in North America are blessed with strong and vibrant intellectual institutions. Particularly in mathematics and the physical sciences, we are flush with good fortune. But success can become a breeding ground for complacency and tunnel vision; there are other countries and communities that have created powerful and successful mathematical and scientific institutions using vastly different formulas than have we. It is in our interests as students of science to find, understand and connect with such communities. This is not so simple as collecting textbooks or hiring faculty. Rather we must understand deeply what makes these scientific communities work, and then adapt that information to improve our own institutions. Open your minds, broaden your horizons, and have a great time on a semester in Budapest. ■

Hex-a-decimal

— Sandy Keith
St. Cloud State University

BARBIE GREGORY and MATTHEW FEIG

Szerda to Vasárnap

We are two Saint Olaf students who attended the Budapest Semesters in Mathematics program in the fall of 1998. Here is a view of students' lives on the BSM program. We hope you enjoy following our character Brett around Budapest from Wednesday to Sunday, Szerda to Vasárnap.

Szerda

"Professor Tóth, I didn't understand your example in lecture this morning. Actually, I'm having trouble understanding geometric probabilities. Could you explain that graph you drew again?" My Probability classmate Jill is asking a question on all of our minds, so we all stuck around for the office hour. Of the four hours of class each week, one is an office hour where we can get lecture and homework questions answered. It's held here in the classroom since the professors' offices are scattered around the city. Now that he explains it again, geometric probabilities are starting to make more sense. I think he's answered my question; I'll go somewhere else to work through these examples again before lunch. Wait, I won't leave yet. Chris is asking about a homework problem, and I'd like to know if I have been on the right track with it.

Wednesdays are great if only for eating lunch at Siesta. This is one of the many restaurants in Budapest where you can get a good sit-down meal for approximately 1000 forints (more or less 5 US dollars), and it's just down the street from the school. Anne and I have been here so often that the waitress already knows what we want to drink. She brings a Fanta and a cola when she sees us. Dining in Hungary is far more leisurely than eating out in the States: if we're not careful to catch our server's attention when she clears our dishes, we might sit at our table for quite a while before we see the bill.

Now it's back to College International. Common practice on Wednesday afternoons is to gather before language class to finish homework and ask each other for study help. I will study for the vocabulary quiz we were promised.

In language class today, the professor is teaching an Hungarian folk song. She presents a variety of activities to keep the learning interesting. We have played board games, practiced rote memorization drills, learned songs and poems, and even translated Winnie the Pooh. This language class is an extension of the two-week intensive course offered before the semester; one of the same professors teaches it. Though I'm not as proficient as I could be, I have become more confident in the Hungarian language. Besides, the language, history, and Hungarian culture courses are nice complements to a morning of mathematics.

It's Wednesday evening, and that means one thing, a home-cooked dinner at Sarah, Jess, and Laura's apartment followed by a group study session. Quality restaurants like Siesta can be found all over Budapest, but you can't eat out every meal, and quite frankly my cooking skills leave a lot to be desired. Group dinners offer a relaxing and fun solution. Problem solving — it's what the BSM program is all about.

It's also what everyone turns their attention to once we've eaten and cleared away the dishes. Almost every BSM math course has one homework assignment due per week, and Thursday is the day for many courses. So all six of us at the apartment have plenty of work to do. Personally, my Combinatorics and Conjecture & Proof problem sets are due tomorrow. Most people are working on Combo, but not everyone. As usual, there are a couple set theory students off to the side debating whether or not a certain theorem depends on the Axiom of Choice. I got a head start on the Combo homework over the weekend, so Conjecture & Proof is my top priority for the night. "What do you guys think about this week's challenge problem for C&P?" I throw out. That promptly set off a half-hour discussion about tiling a chessboard. Each of us had put some thought into the problem before tonight, and after only ten minutes we thought we had an essentially correct solution. Ironing out the details of the proof and convincing everyone that the proof does indeed work, however, takes a while.

With that problem done, our attention turns back to Combinatorics. I'm writing up my solutions to the problems; Sarah and Justin are working through them for the first time. "Brett, how'd you do Combo number five?" they ask a little bit later. I start to explain my answer to them, but while doing so realized I'd overlooked a crucial point. The solution isn't right, but it isn't hopeless either. None of the changes are dif-

BARBIE GREGORY and **MATTHEW FEIG** are mathematics majors at St. Olaf College.

BSM students by the Imre Nagy statue. Nagy gave his life in an effort to liberalize the old Communist regime.

ficult to make, and together the three of us figure it out pretty quickly. The problem makes a lot more sense to me now than it did before. Glancing at my watch, I see that three hours have slipped by since dinner ended. "Good thing you two brought that one up. This is much better than what I had down before. Actually, that was the only problem I had left from the Combo assignment, so I'm going to head back to my place." Walking out to a hail of good-byes and see-ya-tomorrows, I begin to think ahead to the Conjecture & Proof homework I still have to do before I go to bed.

Csütörtök

I am not a fan of morning commutes. During the 7:30 rush hour, the trams and buses are packed. Today class starts at 10:15, so it's not as bad. At this time of day I can often find a seat to read a book or even do some last-minute studying. I take the number 49 bus in to Moszkva Tér ("Moscow Square") where I catch the 61 tram, which drops me off right in front of the school. It's usually around a half-hour commute. Most people have much shorter commutes; many are within walking distance of the school. There is public transportation to any corner of town, even the wonderful Buda Hills hiking trails at the end of the 56 line. From a nearby seat, I overhear someone speaking English on a cellular phone. It always strikes me as odd to hear English from strangers in this town. I wonder how strange it will feel to be surrounded by English back in the States?

I walk into the school building and jog up the steps to the *harmadik emelet*. That's the fourth floor to an American, but the third to a European. All the BSM classrooms and offices are located on this floor. Combinatorics, the largest course in the program, leads off my Thursdays. Professor Dezso makes sure everyone is on the same page before moving to the next topic by asking his trademark questions, "Iz it clear? Well, iz it?" Combo is interesting, but by the time class ends, one thought has gradually taken over and pushed aside any mathematical considerations: must find food ... now. There's a fifteen minute break between classes which is plenty of time to stop by the snack counter downstairs. Coffee and a pastry are exactly what I need right now. When BSM students have to eat on the run, they come down here. There are also several food stands at nearby Móricz Zsigmond Körter, the main transportation hub of south Buda.

Time to go back upstairs for Conjecture & Proof class. Homework's due today, and all two hours of class time are set aside for students to present solutions to the week's problems. For most problems, two or three different solutions come out. Occasionally, though, no one has a proof, or at least no one is confident enough in his or her proof to go up in front of class. Right now, Jess is presenting our proof for the challenge problem. Class is almost over, but Professor Elekes finds just enough time to comment on the proof. Then, as usual, he goes on to show us a much smoother and more elegant way to obtain the same result. It's something you get used to in C&P. Class ends and I leave quickly to go check e-mail, but not before picking up next week's assignment.

Because the demand on the computers is so high, we have to sign up for half-hour time slots in advance. When you only have a half hour, every minute counts, so whenever I get on a computer I go straight to telnet to check my e-mail. E-mail is most BSM students' lifeline to home and college. Occasionally, letters arrive and it's a special treat when they do, but e-mail is the best way to stay in touch on a day-to-day basis. I have one message from my parents, which I expected, but also one from a friend John at my home college. It sounds like things are normal at home, but John has big news: the play he's in is opening this weekend. I wish I could go. At times like this, I become aware of the trade-offs involved in studying abroad. Still, I have never thought, even for a second, that I'd made the wrong choice in coming to Budapest. Quickly, I fire off responses to both John and my family and make way for the person who has the 2:30 slot.

The rest of my afternoon should be pretty low key. I think I'll break up the trip back to my apartment by stopping at a coffee shop and doing a little Probability homework. I don't have much left to do, but I won't have a chance to work on it tonight. I'm going to the opera with my roommate Josh. I have already been to two operas here in Budapest, but this will be Josh's first. The operas are high quality productions and a great way to experience Budapest's cultural atmosphere. The fabulous old opera

S. I. B. GRAY

Beyond Wonderland: The Mathematics of Lewis Carroll

Most students of mathematics know that the mathematician Charles Lutwidge Dodgson of Oxford University wrote *Alice's Adventures in Wonderland* under the pseudonym of Lewis Carroll. Yet very few of us, even the professional mathematicians, can discuss with any specificity what mathematics he investigated. What was the focus of his life's work? What were his mathematical tastes and enthusiasms? Where did this wonderful writer of children's fantasy constructed with a knowledge of formal logic choose to put his professional time? This past year, the centenary of his death, Oxford, Harvard, Princeton, the Pierpont-Morgan Library and others have staged exhibits to honor his life. An examination of his mathematical work provides an interesting glimpse of mathematical study in Victorian England, and tells us a bit about Dodgson too.

Scholars leave evidence of their contributions through publications. Dodgson is no exception. Moreover, Dodgson was a prolific letter writer. Because of his tremendous popularity, much of his correspondence has been preserved in libraries across the U.S. and in Britain. We quote from one of these letters written to his sister, Mary.

February 20, 1861

...As you ask about my mathematical books I will give you a list of my 'Works.'

1. Syllabus, etc., etc., (done)
2. Notes on Euclid (done)
3. Ditto on Algebra (done — will be out this week, I hope)
4. Cycle of examples, Pure Mathematics (about done)
5. Collection of formulæ (done)
6. Collection of Symbols (begun)
7. Algebraical Geometry in 4 vols. (about 1/4 of Vol. I done).

Doesn't it look grand? ... My small friends the Liddells are all in the measles just now. I met them yesterday. Alice had been pronounced as commencing, and looked awfully melancholy—it was almost impossible to make her smile."

Your very affectionate brother,

Charles L. Dodgson

Photograph of Lewis Carroll taken March 28, 1863 by Oscar G. Rejlander.

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From a private collection.

Alice Liddell dressed as a "Beggar Maid," taken by Lewis Carroll, Summer 1858.

One must remember that e-mail, fax, telephone and the automobile were yet to be invented. Communication was slow and difficult. It was not uncommon for family members to exchange detailed letters. We clearly see that as a 28-year-old junior mathematics tutor, Dodgson was hard at work preparing materials for his students. The students in his charge would be examined on the topics in his letter. Then, as now, a student passing a single set of comprehensive exams at the end of a third year of study was the equivalent of a U.S. bachelor's degree. Moreover, under usual conditions, the exams are *only in mathematics.* Each June students still dress in black academic robes to "sit the exams." Under the British system, a tutor works with students in his charge who "read maths." Often an individual student and his tutor form an alliance to prepare for these exams. Ego and pride are at stake. Each student's name and grade is posted on the college bulletin board. Colleges count the number of "firsts" as evidence of the quality of their instruction. Dodgson was writing materials designed to prepare his students for the exams. Without word-processing and photocopying, the University would often print materials for faculty to give to their students.

All of the Works listed in Dodgson's letter to Mary are study guides for his students. The first four are identifiable: they are *A Syllabus of Plane Algebraical Geometry* (1860); *Notes on the First Two Books of Euclid* (also 1860); *Notes on the First Part of Algebra* (1861); *General List of (Mathematical) Subjects, and Cycle for Working Examples* (1863). Most likely the last three are *The Formulæ of Plane Trigonometry* (1861), *Symbols and Abbreviations for Euclid* (1866), and *The Fifth Book of Euclid Treated Algebraically* (1868). Thus, this letter suggests that to major in mathematics at Oxford in the 1860s, the decade of the American Civil War, a student had to know College Algebra, Trigonometry and Euclidean geometry.

His letter also mentions his affection for the Liddell family and especially Alice. Dodgson was the third of eleven children and enjoyed the busy and full life of a rector's son. As an unmarried resident of Tom Quad at Christ Church College, Dodgson enjoyed the friendship of his Dean's family. The head of Christ Church is called a Dean, and Alice was his daughter. At this time, Dodgson would often take the children punting on the nearby Thames or Cherwell. This gave him a chance to enjoy a large family similar to his own, to feel he was helping the Liddells, and to escape the confinement of living, eating, and teaching in the same location. Most importantly, while rowing the children on a quiet river, he honed the storytelling skills that brought him lasting fame.

Dodgson and Euclid

The majority of Dodgson's publications focus on Euclid's *Elements*, especially Books I, II, and V. His *Enunciations* is a study guide to Books I and II of the *Elements*. Dodgson was extremely meticulous, a stickler for both detail and organization. He opened the study of Euclid by asking his students to remember more than thirty definitions, including those of "point" and "line." David Hilbert's axiomatic treatment of geometry establishing the modern view of taking these terms as undefined would be published 40 years later in the classic *Grundlagen der Geometrie*. After the definitions, the students are quizzed on the Axioms:

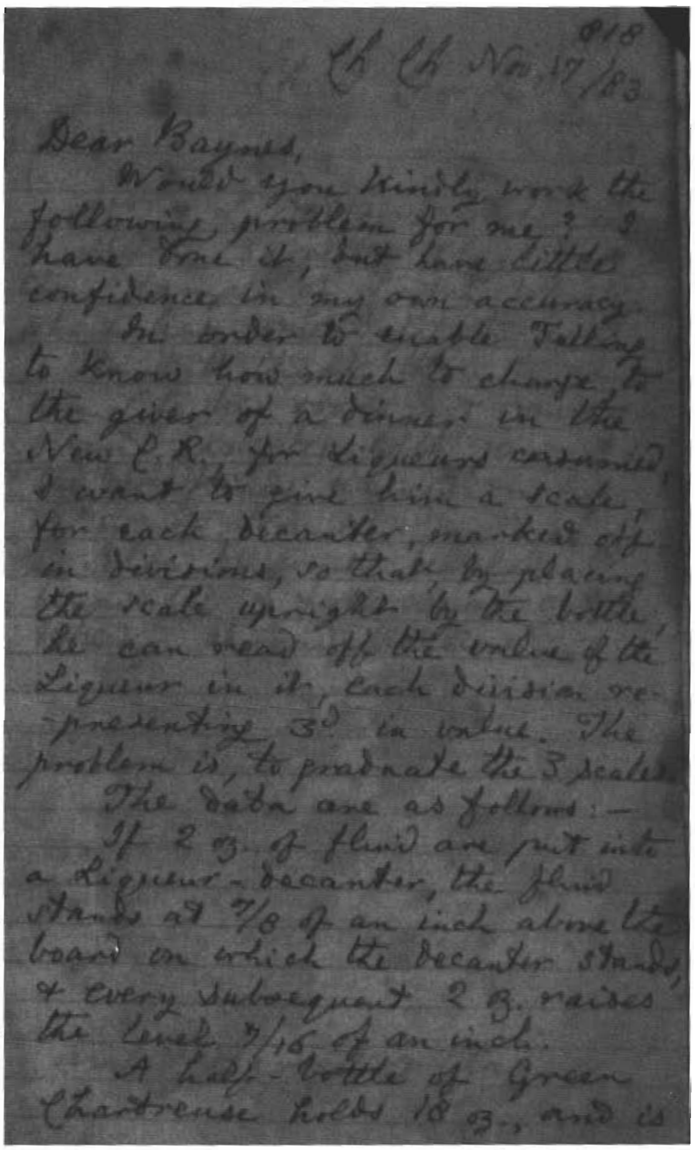
Write out the Axiom concerning:

41. Things equal to the same thing.
44. Magnitudes which coincide with one another.
45. Under what circumstances does Euclid assert, as an Axiom, that two Lines will meet?

Next, Dodgson asks for more definitions, including theorem, postulate, axiom, corollary, (self-referentially) definition, and:

62. Define "A fortiori." Give an instance.
63. Define "Reductio ad absurdum." Give an instance....
72. What are "the complements of the Parallelograms about the diagonal of a Parallelogram"?
73. When are Propositions said to be "converse to each other"? Give an instance.

Note that 44 uses the word "coincide" and thus is an indirect reference to the debate on congruence and transposi tion. Also note that 45 alludes to the controversy over Playfair's Axiom on parallel lines which would lead to Dodgson's future play, *Euclid and His Modern Rivals*. His play, set in Hell, is a dialogue between Minos, who speaks for Euclid, and Niemand, who speaks for those modern geometers who would alter Euclid's axiomatic system. The play strongly, and often amusingly, makes the case that no modern geometry textbook had succeeded in improving upon Euclid. For Dodgson, there could be no exchange of ideas without agreement on a complete set of axioms. He did not accept the non-Euclidean parallel postulate and therefore the discussion was finished as far as he was concerned. For Dodgson truth was based on facts, not theory. "Propositions" were of two kinds: "Problems" for which something is to be done, and "Theorems" for which something is to be believed. Both were to have a proof. Moreover, the proof should have a very particular structure.



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Ch Ch Nov. 17/83

Dear Baynes,

Would you kindly work the following problem for me? I have done it, but have little confidence in my own accuracy.

In order to enable Telling to know how much to charge to the giver of a dinner in the New C.R. for liqueurs consumed, I want to give him a scale for each decanter, marked off in divisions, so that by placing the scale upright by the bottle he can read off the value of the liqueur in it, each division representing 3d in value. The problem is, to graduate the 3 scales.

The data are as follows:—

If 2 oz. of fluid are put into a Liqueur-decanter, the fluid stands at 7/8 of an inch above the board on which the decanter stands, & every subsequent 2 oz. raises the level 7/16 of an inch.

A half-bottle of Green Chartreuse holds 18 oz., and is

Photo courtesy of S. I. B. Gray.

Dodgson was Curator of Senior Common Room, in charge of ordering the wines. Here he poses a measurement problem relating to his duties.

Euclid I, 6

I. General Enunciation.

(1) Data.

If two angles of a triangle be equal to each other,

(2) Quæsita.

the sides also which subtend the equal angles, shall be equal to each other.

II. Particular Enunciation.

(1) Data.

Let ABC be a Δ having

$\angle$ ABC = $\angle$ ACB,

(2) Quæsita.

then side AB shall = side AC.

III. Construction.

For AB $\neq$ AC, one of them is > the other;

let AB be > AC;

from BA cut off BD = AC, and join DC.

IV. Proof.

Then, in Δs DBC, ABC,

$\because$ DB = AC, and BC is common, and $\angle$ DBC = $\angle$ ABC;

$\therefore$ base DC = base AB,

and Δ DBC = Δ ABC,

the < = the >, which is absurd;

$\therefore$ AB is not $\neq$ AC;

V. Particular Conclusion.

that is, AB = AC.

VI. General Conclusion.

Therefore, if two angles, &c.

In this format, the student was encouraged to cover either the data or quæsita with his hand and then "test for himself the accuracy of his recollection of them." In these days an Oxford Fellow and his students would be expected to know both Greek and Latin. The materials for students on what to know and how to write a proof, also contained a "flow chart" illustrating the logical dependencies among Euclid's axioms and theorems. Dodgson and most of his contemporaries in Great Britain were deeply concerned with structure, but only the structure of geometry. Evidence of placing algebra within the framework of a structural system, thus paving the way for 20th century abstract algebra, was only beginning to emerge in the work of Dodgson's contemporaries. In his time, both the Tripos exams at Cambridge and the Honours exams at Oxford emphasized classical geometry.

Dodgson and Linear Algebra

In his 1867 book on determinants, *An Elementary Treatise on Determinants with their application to Simultaneous Linear Equations and Algebraical Geometry*, Dodgson introduced a variation on what is called Cramer's Rule today. While Dodgson's treatment of simultaneous linear equations is strongly suggestive of Cramer, it does feature an unique approach. Citing an example from his book, we read:

$$\frac{X_1}{D_1} = \frac{-X_2}{D_2} = \&\text{c.} = \frac{(-1)^n}{V}$$

where n is the number of equations and V is the block formed by the coefficients of the variables.

Thus, for

$$2x_1 + 3x_2 + 1 = 0$$
$$5x_1 - x_2 - 2 = 0$$

we have

$$\frac{x_1}{\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 1 \\ 5 & -2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix}}$$

That is,

$$\frac{x_1}{-5} = \frac{-x_2}{-9} = \frac{1}{-17};$$
$$x_1 = \frac{5}{17} \text{ and } x_2 = -\frac{9}{17}$$

Cramer's famous treatise *Introduction à l'Analyse des Lignes Courbes algébriques* was published in French in 1750, but is never mentioned by Dodgson, though Dodgson calculates determinants of third and higher orders using coefficients and minors. But then Cramer himself never mentioned the earlier work on determinants now found in Leibniz's notebooks though their notation is similar. Dodgson worked in a solitary fashion and rarely acknowledged any sources from others. At the same time, he was always careful never to claim the work of another as his own.

In 1866, the year following the publication of *Alice* at Christmas time, Dodgson presented his determinant work to the venerable Royal Society. The Royal Society at that time was primarily focused on the controversy brought about by Darwin's *Origin of the Species* published in 1859. Dodgson was never made a member. Also, Dodgson never joined the London Mathematical Society. There is little evidence of interaction between Dodgson and the leading British mathematicians of the day, J.J. Sylvester, Arthur Cayley, Augustus De Morgan, or George Boole.

Success of Alice's Adventures in Wonderland

In the very same decade of his life that he published his works on Euclid and linear algebra, Dodgson wrote *Alice's Adventures in Wonderland*. The widely read *John Bull* news magazine published the first review of the future classic in 1865 and indicates its immediate success. We also find clues as to the 19th-century popular view of mathematicians and to a difference in atmosphere at Oxford and Cambridge.

"The above is *facile princeps* [easily the best] of the Christmas (children's) Books that have come before us during the present prolific season, and fairly deserves a notice to itself. It is quite a work of genius, and a literary study; for, if the reputed author be the true one (and Lewis Carroll is of course a nom de plume), it effectively dispels the notion that first-rate mathematical talent and ability are inconsistent with genuine humour and imagination. Some have so accounted for the differential qualities of the Cambridge and the Oxford character; and it may be, after all, that the prevalent classical atmosphere of the latter University, habitually breathed by our author, has neutralized somewhat the unpoetical fumes of his own special lucubrations, and placed him in such strong contrast with the hard exact-scientific mind of, say, Bishop Colenso. Certainly a more genial and exuberant fancy seldom ranged through dreamland than the narrator of 'Alice's Adventures.'"

The reviewer said he laughed aloud and really liked the book. But mathematicians and logicians can read Alice from an entirely different perspective. For example, from Alice's famous fall, she mutters to Dinah, the cat, "There are no mice in the air, I'm afraid, but you might catch a bat, and that's like a mouse, you know. But do cats eat bats... Do bats eat cats?" Philosophers viewed this as a test of equivalence relations. Eating is not a symmetric relation.

Reading His Letters

In the libraries at Oxford University we find reams of Dodgson's letters to browse. Included are many on the small details of fair procedure. In particular, we find a "POSTSCRIPT," addressed to "Mathematicians only," explaining how he developed an equitable election rotation among the colleges for the coveted Proctorships. As head of the Senior Common Room, i.e., the faculty club, he wrote many individuals concerning fair payment for the alcohol ordered, shipped, or consumed.

Unfortunately we also find that he campaigned against admitting women to the Oxford colleges.

"... I have heard him (the late Dr. Liddon) express, most warmly and earnestly, his fears as to the effect that the new movement, for flooding Oxford with young Women-Students, would have on the young Women themselves. And I have no doubt that, were he yet among us, his silvery tones would have been heard in Congregation last Tuesday, deprecating the introduction, into our ancient University, of that social monster, the 'He-Woman.'"

Dodgson, and all the Christ Church faculty, ate together in "Hall." Students dining on campus today can, no doubt, identify with Dodgson's displeasure of the quality of the cuisine. Toward the end of his career when he was elected to head the Senior Common Room at Christ Church, he complained to the Steward. As might be expected of a great writer, he found an arsenal of words to score his point.

To Michael Sadler
April 14, 1887

Dear Steward,

... (John Henry Onions) and I are agreed ... that it is about time to make a formal representation to you as to the very inferior cookery now prevalent. During the last 10 days or so we have had
(a) Beefsteak almost too tough to eat.
(b) Mashed potatoes that were a mere sop.
(c) Portugal onions quite underboiled and uneatable.
(d) Yesterday I ordered (for the last time: I shall not again) baked apple dumplings. Their idea of that dish seems to be this: "take some apples: wrap each in the thinnest possible piece of pastry: bake till nearly black, so as to produce the consistency of — say pasteboard."
(e) Cauliflowers are always sent with no part soft enough to eat except the tops of the flowers. This the Cook defends, and seems to think no one ever expects to eat more: he explains that, if boiled till the stalks are eatable, the flower would be overboiled. All I know is that everywhere, except here, cauliflower is a very nice vegetable, and eatable as a whole. Here only 5% is eatable, and that absolutely flavorless.
(f) Potatoes (boiled) are never "mealy" as cooked here.
However, these last two are chronic grievances. We never get boiled potatoes, or cauliflowers, properly cooked in Hall.

I don't think I'm remarkably fastidious as to cookery, but I may say that I should think very poorly of a London restaurant (with dinners, say, at 2 s. a head) that supplied such cookery.

Hoping all this won't bother you very much, I am

Very truly yours,
C. L. Dodgson

Courtesy of Junior Common Room, Christ Church.

Odds and Ends

In addition to his letters, the libraries at Oxford have a marvelous collection of Charles Dodgson's memorabilia under lock and key. Fortunately, much has been reproduced. In the Bodleian Library we find leaflets on games, circular billiards, Christmas wishes, Easter greetings, alphabet codes, "on Catching Colds," etc. We find Easter and Christmas greetings for "every child who loves Alice." We also find that the Fellows and students of the colleges placed small bets on puzzles much as we bet on football pools today. Wager books were often found in the dining halls. Carroll created a word puzzle for each week of the term. This is Carroll's example taken from his rules for playing a word game called DOUBLETS: Turn HEAD into TAIL:

HEAD
HEAL
TEAL
TELL
TALL
TAIL

Dodgson was a much admired amateur photographer. He took many pictures of friends, children, relatives, Alice Liddell as a child and an older woman, etc. He wrote poetry. This acrostic verse has the second letters read downwards spell out the name of a pupil, Edith Rix. It appeared as the Dedication of his *A Tangled Tale*, a collection of problems from Dodgson's monthly puzzle column in *The Monthly Packet* magazine.

To My Pupil
Beloved Pupil! Tamed by thee,
Addish-, Subtrac-, Multipli-cation,
Division, Fractions, Rule of Three,
Attest thy deft manipulation!

Then onward! Let the voice of Fame
From Age to Age repeat thy story,
Till thou hast won thyself a name
Exceeding even Euclid's glory.

Dodgson suffered from two major handicaps. There are eyewitness accounts at Christ Church of his suffering epileptic-type seizures. Moreover, he had a very bad speech problem. Indeed, he became a mathematician, rather than a vicar as so many in his family had been, because of his speech problem. On June 19, 1872 he wrote a poignant note to Alfred Tennyson (later Lord) at nearby Lincoln College thanking Tennyson for sending him to a Dr. Lewin where Dodgson "learnt his system for the cure of stammering." This may account in part for his habit of working in relative isolation from other leading mathematicians at other colleges and universities. Still he enjoyed close friendships and collaborations with actors, actresses, producers and playwrights in London. A London theater friend wrote, "He was a born story-teller, and if he had not been affected with a slight stutter in the presence of grownups would have been a wonderful actor, his sense of the theatre was extraordinary." In demeanor, his closest friend described him as a "stately whiskered man" who "when he was not wearing academic dress, was always dressed in top-hat and frock-coat."

But mostly, when we browse Dodgson's papers, we find Euclid. He was devoted to the axiomatic system based in geometry that was at the core of 19th-century British mathematics. In instance after instance, Dodgson's mathematics demonstrated highly organized practical problems solved with unusual methods and notation. But his talent and skill in selecting words often brought a lightness and humor to the fundamentally pedantic flavor of his mathematical works. ■

References

[Note: Many of Dodgson's original works have been reprinted in recent years.]

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Szerda to Vasárnap continued from p. 16

and gave us good directions. Just walking from the station to the hotel has gotten us excited about doing some sightseeing, so we're going to dump our bags as quickly as possible and start exploring the town.

Like every European town, Eger boasts a couple great old churches, but there are also several unique attractions here. A single minaret is all that remains from a mosque dating back to the Turkish conquest of eastern Hungary, and an observatory at the top of the library affords visitors an unparalleled view of the town center. All these buildings are located within blocks of one another in the pedestrian-only section of Eger, but hitting four tourist attractions in a single afternoon is tiring no matter how close together they are. Our plan is to relax back at the hotel until evening and then eat dinner along Eger's famous wine-tasting strip. Every winery and restaurant in the Szépasszonyvölgy, literally translated the "Valley of the Beautiful Woman," serves its very own brands of wine. It makes for a unique dining experience and an enjoyable end to the day.

Vasárnap

Morning arrived far too quickly. We're eating breakfast at the hotel's restaurant. Anne, Karen, and Josh look only half awake, and I can't imagine that I look much more energetic. It's probably a good thing that the only activity on tap for today is a trip to Eger castle. We actually stopped by the castle yesterday and did some looking around, but we were too late to catch the last English tour. Our options were to join a tour given in Hungarian or come back today at 10:00. So here we are! The tour is definitely worth it. I really enjoyed exploring the castle on our own yesterday, but it's nice to hear the history and anecdotes behind the buildings.

At 2:30, not much more than a day after arriving in Eger, the four of us are on a train back to Budapest. Today it was my turn to go first at the ticket counter, but each time I use my Hungarian it gets a little easier. I'm looking forward to catching up on my sleep during the train ride. Glancing around the compartment, I see that Anne and Josh have already beaten me to it.

At Keleti train station, the four weary travelers split up. Anne and Karen hop on the number 7 bus while Josh and I take the escalator down to the red metro line. Back in the friendly confines of our flat, we try to cook a rice dish following a friend's recipe. Both of us are satisfied with how it turns out: not bad for our first time. After cleaning up the kitchen, I slump down into my favorite study chair with a sigh and stare out the window ... but only for a minute. Then I straighten up and pull my backpack close to prepare for a night of studying. After all, a new week begins tomorrow. ■

Answers to Jeopardy Questions (pp. 24–26)

What is / who is:

a) Treasure Island
b) nurse shark
c) Maya
d) Descartes
e) 17th
f) The Pirates of Penzance
g) Newton
h) Peter
i) Cardinals or Giants
j) Stratford
k) Philistines
l) cube
m) mean
n) addition
o) obtuse
p) tangent
q) Donna Shalala

ARTHUR BENJAMIN

Math Major Wins College *Jeopardy!*

Here's a quiz for all of you prospective game show contestants out there. Can you answer these questions (or rather question these answers) in less than 5 seconds each?

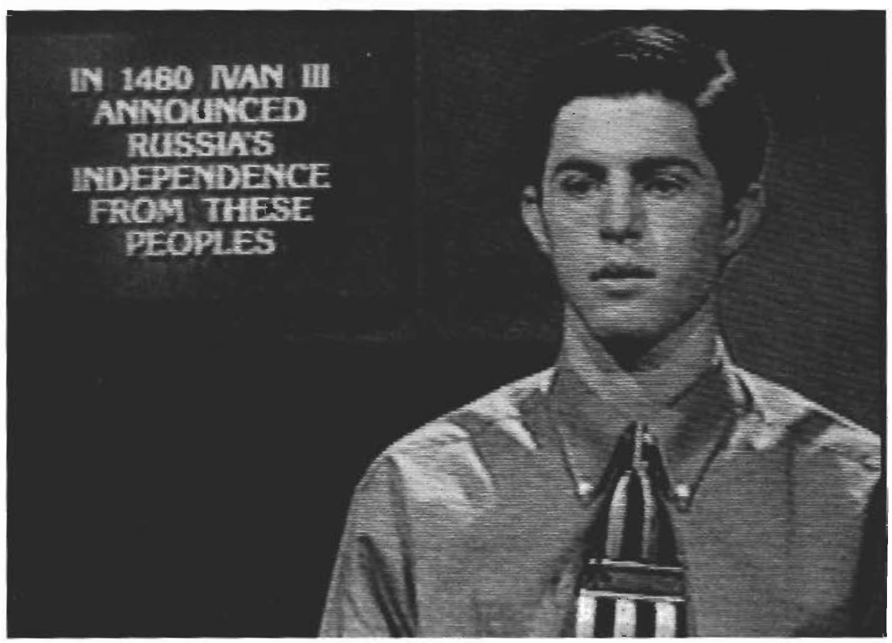
a) The first chapter of this novel is titled "The Old Sea-Dog at the 'Admiral Benbow'"
b) This shark may have earned its name because the male hangs onto the female's fin with its teeth
c) The city of Polenque flourished from 600 to 900 during this North American civilization's Classic Era

(Answers appear on p.23)

Andrew "Rif" Hutchings proved he can answer these and many other trivia questions like a pro in the 1998 *College Jeopardy!* Championship.

Rif, a senior math major from Harvey Mudd College in Claremont, California, became the most recent winner of the *College Jeopardy!* Championship, which aired nationally from May 4 through May 15, 1998. His winnings included $25,000, a new Volvo S70, and the opportunity to appear on *Jeopardy!*'s Tournament of Champions. In his freshman year at Harvey Mudd, Andrew earned the nickname "Rif" (which stands for Random Information Frosh) from upperclassmen who were impressed with his abundant knowledge of trivia. Although his encyclopedic knowledge of facts and trivia was clearly important, it was his knowledge of mathematics and optimal wagering strategy that ultimately secured his victory.

Jeopardy! is a long-running television game show that tests its contestants' knowledge of facts and trivia in various categories, with a twist—the contestants are given the answers and must come up with the questions. To be a contestant for the college tournament, one must first be invited to audition. Invitations are determined by random draw based on postcards sent to the program. Rif, who had watched and studied the program for years while growing up in Tucson, Arizona, sent in 168 postcards over three years, receiving an invitation each year to audition at the Los Angeles studio. There, he and many other aspiring contestants were given a challenging written test. Those with a sufficiently high score were interviewed further and told to wait for a phone call. In his sophomore year, Rif did not score high enough on the test to merit an interview. In his junior year he earned an interview, but did not make the final cut. Then, in his last year of eligibility as a college student, he was chosen to be



Rif wracking his brains.

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house is an attraction in and of itself. For all that, Josh remained unconvinced until I mentioned that he would be passing up the opportunity to see Mozart's *Marriage of Figaro* for the equivalent of about two American dollars. I'd better get on my way if I'm going to fit in homework and cooking before we leave.

Péntek

Probability homework is due this morning. The problem sets are usually challenging but doable. This assignment had the typical mix of straightforward and complicated problems; I'm sure I didn't get them all correct. After today's two hours of lecture, I will hand the assignment in and go play soccer. Justin and I, both soccer fans, have gotten in with a group of Hungarian students who play regularly. Besides soccer, students of BSM take fencing lessons, go folk dancing, and work out at a gym here in Budapest. We have each found ways of pursuing our hobbies, and I can usually find someone interested in joining me; I just have to ask.

The Hungarians play a fierce game of soccer. I'm usually so tired afterwards that I can do little else but go home and shower, eat, and sleep. I'm running low on groceries, but I think I can scrounge together some lunch. I definitely need to make a shopping run this afternoon. My favorite place to shop is the large market over in Pest, but today I think I'll stick to my own neighborhood. The first Western-style mall in Budapest, the Mammut (Mammoth), opened recently. On the lowest floor is an extensive supermarket, and outside there are a number of produce, meat, and cheese stands. In general, the produce in Budapest is good-quality and varied. For lack of alternatives vegetarians tend to eat a lot of fried cheese in restaurants, but when cooking at home it's easy to go "crazy with the veggies," as a fellow BSMer likes to proclaim.

Though it's nice to be mistaken for a native Hungarian, it's always awkward when someone stops you on the street for directions. My Hungarian is generally not good enough to understand more than the basic phrases I use every day; my ear tends to be rather slow. This afternoon, for instance, someone stopped me as I was walking from Moszkva Tér to the Mammut a block away. "Lassabban kérem. Slower, please," I had to ask. I think she was asking me for directions, but even when she repeated it more slowly, I could not understand. Finally, we were both getting frustrated when I said "Sajnos, nem értem," and she walked off. "Sorry, I don't understand" is a phrase altogether more common to my Budapest life than I would like.

Tonight I'm meeting some friends at a restaurant called Saint Jupat's. It is typical Hungarian cooking: most dishes are deep fried, but it's good food, and the servings are huge. After dinner there is a party at a friend's apartment over in Pest. I think I'll stop by for a while, but I have to leave before the bus to my apartment stops running at 11:00. I'm taking a trip to the town of Eger tomorrow, so I want to be sure to get a good night's sleep. Students can easily travel on the weekends to Prague, Vienna, Croatia, Slovenia; we chose Eger to explore some more of Hungary.

Szombat

The plan was to meet at 8:00 a.m. at Keleti Pályaudvar, Budapest's Eastern Railway Station, but as Josh and I look around, we don't see Anne or her roommate Karen. "There they are," Josh suddenly calls out, pointing off to the right, "Let's grab them and get in line for tickets." We have all traveled by train before, so buying tickets should go smoothly. Anne has the toughest job because she's first. Once she figures out where, when, and how much, the rest of us will just mimic the Hungarian she uses. That should leave us with plenty of time to walk to the right train and find a seat. Eger, here we come.

Eger is about a four hour train ride east of Budapest. We pass the time by alternately resting and discussing math problems. Whether eating out, relaxing in someone's apartment or taking the train to Eger, conversations like this happen all the time among the BSM students, and they are a great way to get work accomplished in a relaxed atmosphere. Josh, Ann, and Karen are all in Graph Theory, so that is the hot topic this morning. For a while, I tried unsuccessfully to bring the discussion around to Conjecture & Proof since I'm not in Graph Theory, but I've given up and am just trying to follow along as best I can.

After arriving in a new town, the first order of business is to find a hotel, preferably a cheap one. Luckily, some friends of ours visited Eger a few weeks ago, so they recommended a nice place to us
(continued on p. 23)

Photo by Kawa Kamal.

1999 BSM participants in front of the Castel of Buda.

one of 15 college students (9 female, 6 male) to appear in the tournament.

All segments of the program were taped on the campus of UC Berkeley on March 21 and 22, 1998. To win the tournament, contestants have to survive 3 rounds of competition. Each program consists of three sets of questions: *Jeopardy!*, *Double Jeopardy!* (where the answers are harder and the point values double), and *Final Jeopardy!* (one final question for all contestants), where each contestant may wager any amount up to the total number of points he or she currently has. Rif won his first round, partly due to sweeping the category "All About Calculus" in *Double Jeopardy!*. See how you fare on these:

d) Points on the x and y axes are called Cartesian coordinates in honor of this man
e) It's the century in which calculus was created
f) Calculus is featured in this Gilbert and Sullivan operetta, heard here: "I am very good at integral and differential calculus, I know the scientific names of beings animalculous..."
g) With Cambridge closed because of the plague, he developed calculus on his mother's farm in Woolsthorpe
h) Leibniz, an inventor of calculus, was an advisor to this "great" Russian czar

In *Final Jeopardy!*, Rif held onto his lead by correctly answering this question from the category "Sports Teams":

i) 1 of 2 names shared by a major league baseball team and an NFL team

In his second round (the Semi-Finals), Rif held the lead against his two opponents, Claire (from Yale University) and Grace (from Barton College). Going into *Final Jeopardy!*, the scores were: Rif 10400, Claire 8200, Grace 4600.

The *Final Jeopardy!* category was "Places in Canada." Before the answer is read, contestants are given as much time as they need to determine how much they should wager. They do not know what the others are wagering. The winner may be determined not only by who gets the answer right, but by who bets most skillfully. Before reading further, decide how much each player should bet.

Rif correctly questions the Final Jeopardy answer on the first day of the Finals.

Rif's optimal strategy: Keep in mind that only the winner gets to keep any money (In the College Tournament ties are broken with another question), so each player should simply try to maximize his/her probability of winning. To protect his lead, Rif needs to assume that Claire will bet everything (in fact this is the typical, though rarely optimal strategy of the second player — see the article by Gilbert and Hatcher). Therefore, unless he feels that he is highly unlikely to figure out the "Places in Canada" question, he should bet (8200*2) – 10400 + 1 = 6001 points.

Claire's optimal strategy: Claire should realize that in order for her to win, Rif must miss the question, which will most likely leave him with 10,400 — 6001 = 4399 points. So, to be able to beat Rif when he misses, she should bet no more than 8200 – 4399 = 3801 points. However, to protect her lead from Grace, she should bet at least (4600*2) – 8200 + 1 = 1001 points. In this situation, it would be wise to wager 1001 points, although wagering 2201 would cover the unlikely event that Rif would bet zero.

Grace's optimal strategy: Grace actually has the trickiest decision. Grace should realize that if Rif or Claire gets the question right, then she will almost certainly lose. Thus, she should bet on the assumption that both Rif and Claire will miss. As explained above, Rif would end up with 4399 points. To stay ahead of Rif (even if she misses), Grace needs to bet at most 4600 – (4399 + 1) = 200 points. What does she need to stay ahead of Claire? Rationally, Claire could bet any amount between 1001 and 8200 points. If Claire bets small (i.e., under 3600), Grace needs to get the question right in order to win, so Grace would need to bet at least 3600 points. If Claire bets large (i.e., over 3600), then betting 0 allows Grace to win either way. Therefore, it makes the most sense for Grace to bet 0 (or maybe 100 just to show off) in this situation for three reasons. First, statistically most people in Claire's situation will bet almost everything. Second, betting 0 does not require Grace to get the answer right. Finally, if Rif and Claire both miss the question, which is the only relevant scenario, then it probably was a hard question!

The *Final Jeopardy!* answer was (remember, the category was "Places in Canada"):

j) Renamed in the 1830's, this city of 28,000 and its river were both originally called Little Thames

As luck would have it, all three contestants missed this question! Rif wound up betting 6001, while Claire and Grace bet 8199 and 4599, respectively, leaving them with 1 point apiece. Either could have won the match just by wagering properly. Rif won the match and moved on to the Finals. He later admitted that he knew the proper betting strategy for all three scores.

The *Jeopardy!* Finals consisted of a two-day event (though the two shows are actually taped back-to-back on the same day). The winner is the contestant with the highest two-day total score. Opposing Rif were Alex from MIT, and Shane from the University of Alabama, who dominated the first day until the very last moment. Going into *Final Jeopardy!*, the scores were 9600 for Shane, 7400 for Alex, and 4000 for Rif.

The *Final Jeopardy!* category was "Words from the Bible," and the answer was as follows:

k) A 17th-Century sermon on the book of Judges led to this group's name being applied to uneducated townspeople

Shane, Alex, and Rif bet 7100, 6000, and 4000 respectively. Shane missed the question, but Alex and Rif got it right. At the end of the first day of the Finals, the scores were 2500 for Shane, 13400 for Alex, and 8000 for Rif.

On the second day of the Finals, all contestants started with a score of zero, and they could wager only money earned on that day. In the first round, one of the categories was "Math Vocabulary," in which Rif accumulated 1400 (out of 1500 possible) points. He found all of these questions to be easy (he is a math major after all). See if you can answer them in competitive tournament conditions (about 2 seconds each):

l) Raise a number to the third power, or a common form of ice and sugar
m) The average value of a set of numbers, or cruel
n) The operation of combining numbers, or a wing appended to a building
o) An angle greater than 90 degrees but less than 180 degrees, or lacking sharpness of intellect
p) A trigonometric function that equals sine/cosine, or a conversational digression

Going into *Final Jeopardy!* on the second day, the scores were (not including the first day's totals): Shane with 1000 points, Alex with 7500 points, and Rif with 8800 points. The *Final Jeopardy!* category was "Power People". How much of his 8800 points should Rif bet? After deciding the wager, contemplate the *Final Jeopardy!* answer:

q) Under current succession laws, this former university head is the last woman in line for the US presidency

In dramatic fashion, the finalists revealed their questions and their wagers. Shane wagered 1000 points and came up with the correct question. Alex wagered 4701 points (exactly as expected) but missed the question. Most of the audience was surprised when Rif correctly answered the question, yet had wagered only 300 points. But when the dust settled and the two-day total scores were shown, Shane had 4500 points, Alex had 16199, and Rif had 17100 points, a new car, and $25,000.

As a bonus, Harvey Mudd College also received $25,000 from Volvo to be used for scholarship purposes. At the suggestion of Rif, math department chairman Michael Moody, and Rif's classmate Aaron Archer, the college used the money to establish the Rif Hutchings Endowment. The endowment will fund a scholarship to help attract incoming students of particularly high mathematical ability, and will also reward the three highest scorers at Harvey Mudd in the William Lowell Putnam examination in mathematics. (Rif would certainly have qualified in his sophomore year; he scored 45 points on the Putnam exam and was ranked 107th in the nation.) Through an intensive fund raising effort led by Aaron Archer and propelled by a generous donation from senior Patri Forwalter-Friedman, the endowment has already increased to $75,000 in receipts and pledges.

Rif spent the rest of his Spring immersed in his classes, senior thesis, clinic project, play rehearsals, interviewing graduate schools and summer employers, and preparing for the Toronto math conferences sponsored by SIAM and MAA. In Toronto, Hutchings, Archer, and HMC senior Brian Johnson presented their Mathematical Contest in Modeling paper, for which they received the SIAM Prize last year. Rif also participated in the National Problem Solving Competition (sponsored by MAA), where he finished in second place. Rif began a PhD program last fall at Cornell University, where, appropriately enough, he is studying Operations Research—the mathematics of doing things optimally. ■

Alex Trebek presents Rif with the College Jeopardy! *championship trophy.*

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PATRICK HEADLEY

How I Lost on *Jeopardy!*

When I teach mathematics, I try hard to encourage my students to participate in class. Many of them, however, are just too nervous to risk being wrong in front of me and their friends. So, sometimes, I tell the story of the mistake I made that was seen by millions. Back when I was a graduate student, I appeared on *Jeopardy!*, and, in spite of my mathematical training, I failed to analyze the *Final Jeopardy!* round thoroughly enough. If you are familiar with the game, you may be wondering how this could happen. At first glance the rules seem simple. However, we shall see that even this simple game illustrates the paradoxes and pitfalls of *game theory*, the branch of mathematics devoted to the study of strategy in games.

If you are playing a game and you want to analyze it, you must ask yourself two questions. First, what are your goals? In *Jeopardy!* players have one main goal: winning the most points and qualifying to play again. An outright win is slightly preferable to a tie, since it prevents an able, experienced player from continuing with you to the next game. The next question is, what kind of strategy do you expect from your opponents? One way to answer this question is to gather data from many games and see what players tend to do. Gilbert and Hatcher [1] did this (their article is a more technical look at *Jeopardy!* betting). Here is a somewhat simplified summary of what they found. The first-place player going into *Final Jeopardy!* usually bets just enough to guarantee an outright win with a correct question, no matter what the others players do. The second- and third-place players tend to bet aggressively, often betting all of their points.

How should you bet in response to opponents playing this way? Assume player 1 has A dollars, player 2 has B dollars, and player 3 has C dollars. To simplify things for the moment we will assume there are no ties, and $A > B > C$. We will also disregard certain special cases. If you are player 1, your opponents will either double their bets or lose everything. If $A > 2B$, you can bet nothing and have a sure win. If $A < 2B$ and $A < B + C$, the only thing to do is guarantee a total over $2B$ with a correct question by betting at least $2B - A + 1$ points. If $A < 2B$ and $A > B + C$, you should still bet at least $2B - A + 1$ points, but you can simultaneously guarantee a win against player 3 by betting at most $A - 2C - 1$ points, since losing this many points leaves you with a total of $2C + 1$.

If you are player 2, you have to hope that player 1's response is wrong, leaving player 1 with $2A - 2B - 1$ points. You certainly have to try to reach this number, but you also have to worry about player 3 reaching $2C$ with a correct question. If $2A - 2B - 1 > 2C$, your strategy is clear: you might as well bet nothing if you already have more than $2A - 2B - 1$ points, but, if $B < 2A - 2B - 1$, you have nothing to lose by betting everything you have. On the other hand, if $2C > 2A - 2B - 1$, the situation is more complicated. The data provided by Gilbert and Hatcher [1] show that there are few games in which player 1's question is wrong while player 3's is right. So you should not take much of a risk in order to top $2C$. If $B > 2C > 2A - 2B - 1$, bet nothing. If $2C > 2A - 2B - 1 > B$, you might as well bet it all and hope for a correct question, since an incorrect one means a certain loss. Finally, if $2C > B > 2A - 2B - 1$, and B is closer to $2C$ than $2A - 2B - 1$, then you can bet $2C - B + 1$ to go after a total of $2C + 1$ without risking falling below $2A - 2B - 1$. But if B is closer to $2A - 2B - 1$ than to $2C$, you can't do this. It would be appropriate to bet nothing and stay over $2A - 2B - 1$.

Finally, if you are player 3, your strategy is simple since there are so few ways to win. Both players 1 and 2 have to lose their bets. Assuming this leaves player 1 with $2A - 2B - 1$ and player 2 with nothing, the $2A - 2B - 1$ level must be your goal. If $C > 2A - 2B - 1$, you can bet nothing. If $C < 2A - 2B - 1$, you have nothing to lose by betting it all.

The strategies I have outlined here have one obvious flaw: they wouldn't be very helpful if everyone used them. To illustrate this, imagine that Kara and Ian play two-player *Final Jeopardy!* repeatedly. Kara always begins with 10000 points and Ian with 8000 points. They have both read the preceding paragraphs, so Kara bets 6001 points and Ian bets nothing.

	Kara	Ian
Final score with correct question	16001	8000
Final score with incorrect question	3999	8000

After a while, Kara notices that Ian always bets nothing, and she can change her bet to 1000 to guarantee victory:

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	Kara	Ian
Final score with correct question	11000	8000
Final score with incorrect question	9000	8000

Once Ian catches on, he changes his bet to 4000 and can now beat Kara even if her question is correct:

	Kara	Ian
Final score with correct question	11000	12000
Final score with incorrect question	9000	4000

Where will this end? It doesn't. That is, if each player knew the other's bet, at least one would always change his/her own bet if given the chance. In the language of game theory, we say that the game has no *equilibrium point*.

Given this, we can expect that Kara and Ian will try to change their bets from game to game in an unpredictable way, although probably still favoring some choices over others. What if Ian starts to realize that Kara is doing a good job of anticipating his bets? He can respond by making his decisions dependent on something impossible to anticipate, such as coin-flips or dice-rolls. This kind of strategy is a *mixed strategy*, in which the player picks several options, decides the probability that each option will be used, and then makes a choice at random based on these probabilities.

It seems very strange that Ian might want to leave his decisions to chance. But it puts Kara in a difficult situation even if she knows Ian's strategy in detail. Her best response is to find a mixed strategy of her own that often defeats Ian's without giving him much incentive to adapt his strategy in return. In fact, when mixed strategies are considered, there definitely *will* be at least one equilibrium point. At this point, Kara and Ian could reveal their mixed strategies to each other, and neither could improve their position if allowed to change strategies unilaterally.

The existence of mixed-strategy equilibrium points in games such as this was established by John Nash in 1951 [3]. Nash won a Nobel Prize for his work in games [2], which is surprising since there is no Nobel Prize for mathematics! Instead, his prize was for economics. The ideas used to analyze games are also used to study the decisions made by businesses and consumers. See [4] for an excellent collection of applications of game theory.

Unfortunately, Kara and Ian have so many possible wagers to consider that actually computing equilibrium points isn't very practical. The equilibrium points also have a surprising flaw. Returning to Kara and Ian's game, assume that Kara bets nothing and Ian bets 2000.

	Kara	Ian
Final score with correct question	10000	10000
Final score with incorrect question	10000	6000

This is not an equilibrium point, since Ian has nothing to lose by betting more and going for the outright win. It is, however, a very good situation for both players, since both players can be happy with a tie. Kara is guaranteed $10000 and at least a tie. If Ian's question is correct, he also wins $10000 with a tie.

For both players, this is a better result than they could hope for with an equilibrium-point strategy. The difference is that this scenario depends on cooperation (or conspiracy, depending on your point of view) between the two players. If Kara and Ian trusted one another and were allowed to discuss their bets with each other, they might agree to make these bets. It's quite unlikely that they would bet this way without any communication. Even with an agreement, one player might choose to go back on his/her word, either to be purposely deceptive or through a lack of trust in the other player's motives.

This sounds as much like a discussion of the Cold War or international trade as it does of *Final Jeopardy!*. What all of these situations have in common is that a gain for one player does not necessarily mean a loss for another. This often results in a situation where cooperation has some value but also carries some risk. It also may mean that the self-interest of one player naturally leads to an opportunity for another. This is exactly what happened to me when I appeared on *Jeopardy!*.

Going into *Final Jeopardy!* I had 4700 points. I was in second place, behind Shane with 6900 points but ahead of Andrea with 2200 points. (This is a special case left out of the analysis earlier, since $A = B + C$.) I assumed that Shane would do the usual thing and bet 2501 points, thereby guaranteeing a win with the correct question. So I bet 200 points, knowing I would be guaranteed a win if Shane was wrong.

If I had put myself in Shane's shoes for a moment, I would have seen things differently. While 2501 was a good bet for defeating me, it was a bad bet for defeating Andrea. Losing 2501 points would leave Shane with 4399; Andrea could double her 2200 and beat Shane by a single point. So Shane bet 2500 points exactly. If he was right, $9400 and a spot on the next show were assured. If he was wrong, Andrea at least could not overtake him.

Here was my opportunity. If I had seen that Shane had a good reason to allow for a tie, I could have bet everything I had. As it turned out, we both won our bets, so I could have tied him at 9400 points. Instead, I was a distant second, puzzling over Shane's bet. It wasn't until hours later, after I had returned to my Hollywood hotel room, that I understood what had happened.

Of course, betting everything still would have taken some daring. Would I have done it? I don't know. I just wish I had thought of it quickly enough to consider it, especially given my choice of career. Fortunately, my appearance on *Jeopardy!* was far from a complete loss. The second-place prize was a trip to Florida and the Bahamas. Even better, a conversation with an acquaintance about my "thirty minutes of fame" led to one thing and then another, and that acquaintance is now my wife! So I can't complain. I'll just always be a little embarrassed about my $9400 mistake. ■

References

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3. John F. Nash, Non-cooperative games, *Annals of Mathematics* (2) 54 (1951), 286–295.
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JEFFREY ONDICH

Books for the Beach

If you're looking for something to help you while away those idle hours at the beach this summer, you might want to consider a math book. And while books like Halmos' *Naive Set Theory* and Spivak's *Calculus on Manifolds* are elegant little volumes that would make a different sort of fashion statement than would the latest in swimwear, they are, after all, textbooks, from which you might reasonably desire a break. Instead, try one of the many books of mathematical history, biography, puzzles, and appreciation. Here is a list of some good ones to get you started. Most of them are even in print. (I have listed the dates of the most recent printings I could find.) Happy reading.

Flatland, a Romance of Many Dimensions, by A. Square (a.k.a. Edwin Abbott Abbott), Dover Publications, 1992. First published in 1884, *Flatland* is narrated by A. Square, a mathematician living in a two-dimensional universe. Mr. Square has a special kind of Y2K problem when, on New Year's Day of the year 2000, he is kidnapped by a sphere and transported into the baffling world of three dimensions. Still the best place to start if you want to try to visualize four-dimensional (and higher) geometry, *Flatland* also provides a curious perspective on the foundations of sex roles and class distinctions.

Gödel, Escher, Bach: An Eternal Golden Braid, by Douglas R. Hofstadter, Basic Books, 1999. This enormously ambitious and entertaining book, winner of a 1979 Pulitzer Prize, starts with a bunch of stories about Bach, ends with an assessment of the prospects for artificial intelligence, and along the way visits recursive function theory, decidability, the computational structure of DNA, computer chess, the nature of consciousness, and much more. Everything in the book, including the many odd and delightful dialogues between Achilles, the Tortoise, and their friends, centers around the self-referential "strange loops" that Hofstadter finds lurking in pretty much every human endeavor, including Bach's music and Escher's art.

A Mathematician's Apology, by G. H. Hardy, Cambridge University Press, 1992. (First published in 1940.) Hardy might not have thought much of the other authors on this list, since they are largely concerned with mathematical exposition, which Hardy considered "work for second-rate minds." As maddening as this and other opinions in the *Apology* can be, Hardy has presented both a powerful defense of the intrinsic value of mathematics, and a tragic portrait of himself. The biographical foreword by C.P. Snow in post-1967 editions gives personal details about Hardy and his relationships with Littlewood, Ramanujan, and the game of cricket, and is worth the price of the book all on its own.

Journey Through Genius: The Great Theorems of Mathematics, by William Dunham, Penguin USA, 1991. There are many books dedicated to the analysis and appreciation of great paintings, great symphonies, and great poems. *Journey Through Genius* is a book about mathematical masterpieces. For each of twelve theorems Dunham considers (ranging from Hippocrates' Quadrature of the Lune to a pair of theorems of Cantor), he provides historical context, biographical detail, and a careful, well-commented version of the theorem's proof. This book takes more concentration than many of the others on this list, but it's worth it.

The World of Mathematics, "a small library of the literature of mathematics from A'h-mosè the Scribe to Albert Einstein, presented with commentaries and notes by James Newman," Microsoft Press, 1988. (First published in 1956.) A small library indeed. This four volume set is part history, part biography, part philosophy, and part source book (including Euler's paper on the Königsberg Bridges, Turing's paper on the Turing Test for intelligent machines, Mendel's paper on the mathematics of heredity, and many more). Set an alarm clock before you start reading to make sure you remember to eat.

Men of Mathematics, by Eric Temple Bell, Touchstone Books, 1986. (First published in 1937.) Go dueling with Galois and swashbuckling with Weierstrass! This collection of biographies of great mathematicians is rollicking good fun. It does contain a fair number of facts that are not entirely—how can we say this politely?—well, factual. But if you want to learn the biographical folklore of mathematics, and you're willing to stomach a fair amount of chauvinism—sexual and otherwise—you have to read this book. If you want to know more about Bell himself, try *The Search*

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for E.T. Bell, by Constance Reid, Mathematical Association of America, 1993. And speaking of Constance Reid,…

Hilbert, by Constance Reid, Springer-Verlag, 1996. (First published in 1970.) Reid may be the best of mathematical biographers, and *Hilbert* is one reason why. This story of a great mathematician is told against a background of political and mathematical upheaval. The book has a technical appendix by Hermann Weyl on Hilbert's mathematics, and a collection of photographs, including the famous picture of Hilbert in a big floppy hat, which was "one of a group of portraits of professors which were sold as postcards in Göttingen." Now those were the days, when professors appeared on trading cards!

Anything by Martin Gardner. For twenty-five years, Gardner wrote a monthly column called *Mathematical Games* in *Scientific American* magazine. You might not be able to recreate that feeling of monthly anticipation by reading the books compiled from Gardner's columns, but all the fun is still there, as are Gardner's elegant prose and immaculate mathematical taste. Meet and analyze the game of "Sprouts" in *Mathematical Carnival* (MAA, 1988) or discover the origin of the great public-key cryptographic challenge known as RSA-129 in *Penrose Tiles to Trapdoor Ciphers* (MAA, 1997).

Raymond Smullyan's logic puzzles are fabulously entertaining, but his books reach much further than the already noble goal of intellectual amusement. Gently guiding his readers on adventures to the Island of Knights (who only tell the truth) and Knaves (who only lie), or to "a certain enchanted forest … inhabited by talking birds," he provides relatively non-technical expositions of some very hard and important results in logic. Try *The Riddle of Scheherazade*, Harvest Books, 1998, *The Lady or the Tiger*, Times Books, 1992, or *The Tao is Silent*, Harper San Francisco, 1992.

If that innocent young niece or nephew joins you at the beach, bring along *Math Curse*, by Jon Scieszka and Lane Smith, Viking, 1995, as a distraction (or possibly a math recruitment tool). This story is about an androgynous kid who suddenly starts to notice mathematical structure in everything s/he looks at or thinks about. The pictures are hilarious (check out the hands on the inhabitant of "Planet Binary"), and the predicament should seem familiar to most *Horizons* readers. Try to get a copy with the dust jacket intact, so you can see what may be the only "About the Authors" section ever to feature a Venn diagram.

If you haven't rejected textbooks out of hand, take a look at the MAA's *Carus Monographs* series (some are out of print, but many are not, and in any case, most college libraries have at least a few of them). These small books are intended to present serious mathematics; nevertheless, some are quite accessible. Harry Pollard's *Celestial Mechanics* (which includes a beautiful treatment of the restricted 3-body problem, from which you can find out where the L5 Society hopes to position a space station), Ivan Niven's *Irrational Numbers*, and Charles Livingston's *Knot Theory* are just three of these excellent books.

I have no doubt that I have snubbed some of your favorites, as I have snubbed some of my own. I haven't mentioned John Allen Paulos's *Innumeracy*, Vintage Books, 1990, and *A Mathematician Reads the Newspaper*, Anchor, 1996. Or *Mathematical People: Profiles and Interviews*, edited by Donald J. Albers and G. L. Alexanderson, Contemporary Books, 1985 and *More Mathematical People*, edited by Albers, Alexanderson, and Constance Reid, Academic Press, 1994. Or *Prof. E. McSquared's original fantastic, and highly edifying calculus primer*, H. Swann and John Johnson, W. Kaufmann, 1975; *How to Lie with Statistics*, Darrell Huff, W. W. Norton & Company, 1993; or *How to Solve It*, G. Pólya, Princeton University Press, 1988. Or some of my favorites from physics and computer science: *Mr. Tompkins in Paperback*, by George Gamow, Cambridge University Press, 1993, *The New Turing Omnibus*, by A.K. Dewdney, W.H. Freeman and Co., 1993, and *ACM Turing Award Lectures: The First Twenty Years*, ACM Press, 1987.

If you finish all of these, let me know. I have lots of suggestions. ■

Answers to the February Final Exam, "Who Said It?" Check the Math Horizons web page for the names of the winners.

1. Isaac Newton
2. Pierre-Simon Laplace
3. G. H. Hardy
4. Albert Einstein
5. Alfred North Whitehead
6. Tweedledum and Tweedledee in "Through The Looking Glass and What Alice Found There," by Lewis Carroll
7. Arthur Cayley
8. Bishop Berkeley
9. Paul Erdős
10. Bertrand Russell
11. Ralph Waldo Emerson
12. Bernard le Bovier de Fontenelle
13. Sonia Kovalevskaia
14. W. H. Auden
15. Smilla Jaspersen in "Smilla's Sense of Snow," by Peter Høeg

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Solutions

September 1998

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Problems Proposed

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Problem Section

Editor

Murray Klamkin
University of Alberta

All problems and/or solutions should be submitted in **duplicate** in easily **legible** form (preferably printed) on **separate** sheets containing the contributor's name (if one is a student this should be so noted), mailing address, school affiliation, and sent to the editor, Mathematics Department, University of Alberta, Edmonton, Alberta T6G 2G1, Canada. If an acknowledgement is desired an e-mail address or a stamped self-addressed postcard should be included (no stamp necessary for outside Canada and the US).

"S" designated problems are problems set particularly for secondary school students and/or undergraduates. Problems with a dagger † indicate they are not original. "Quickie" problems, which are not indicated as such, have their immediate solutions at the end of the section and solutions should not be submitted for them unless the solution is more elegant or is a worthwhile new generalization. For problem submission information, see the February or April, 1996 issues.

Proposals

To be considered for publication, solutions to the following problems should be received by September 1, 1999.

S-27. Proposed by the Problem Editor. One day Mr. Smith arrived at his home railway station 65 minutes before the usual time at which his chauffeur would arrive punctually to drive him home. Instead of waiting for his chauffeur, he immediately started walking home at the speed of 4 miles per hour. He was picked up by his car 4 miles from home and driven there, arriving 10 minutes earlier than usual. If the chauffeur always drove at the same constant speed, how long would it take the car to make the trip from Mr. Smith's home to the railway station?

S-28. T. B. W. Spencer, London, U.K. $ABCD$ is an inscribed quadrilateral in a circle such that $AB + BC = CD + DA$. Prove that $AB + BC \leq \sqrt{2}AC$.

S-29. Proposed by E. M. Kaye, Vancouver, B.C. Determine an infinite family of relatively prime integer triples (x, y, z) such that $x + y + z$ exactly divides $x^2 + y^2 + z^2$.

S-30. Proposed by K. S. Murray. Determine the smallest radii of three congruent disks in order that they can cover a disk of radius r.

117. Proposed by Andy Liu, University of Alberta. Each of six friends has a piece of juicy gossip. They are eager to share them with the others. Whenever two of them get on the telephone, the conversation lasts an hour, during which each tells the other everything he knows so far. What is the minimum number of hours before each of them knows all six pieces of gossip, if (i) three pairs of them can engage in telephone conversations independently but simultaneously; (ii) only one pair can engage in a telephone conversation per hour?

118†. A circle intersects a rectangular hyperbola in four points. Prove that the sum of the squares of the distances from the points to the center of the hyperbola equals the square of the diameter of the circle.

119. Proposed by P. Wagner, Chicago, IL. Determine the nth derivative of $\sqrt{x^2 - 1}$.

120†. Two non-intersecting congruent parabolas in a plane have the same axis. Prove that the area bounded by one of the parabolas and a tangent line to the other parabola is constant.

121. Proposed by Andy Liu, University of Alberta. F is a fixed point on the diameter AB of a semicircle, but distinct from the center O. P and Q are variable points on the semicircle such that $\angle PFA = \angle QFB$. Prove that the line PQ passes through a fixed point other than F.

122. Proposed by K. M. Seymour, Toronto, ON. Determine all k such that the sequence $\{x_n\}$ defined by $x_{n+1} = 2x_n + 1 + \sqrt{3x_n^2 + 6x_n + k}$, $n = 0, 1, 2, \ldots, x_0 = 0$, consists solely of integers.

Solutions

Problem 97†. Limit of an Integral

Determine

$$\lim_{n\to\infty} \sqrt{n} \int_0^\pi \sin^n x\,dx.$$

Editorial note. The following solution by Danrun Huang, St. Cloud University had been misfiled and is more self-contained than the previously published solution.

Using the reduction formula

$$\int \sin^n dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx,$$

$n \geq 2$, we have that $A_n = [(n-1)/n]A_{n-2}$ $(n \geq 2)$ where $A_n = \int_0^\pi \sin^n x\,dx$ and $A_0 = \pi$, $A_1 = 2$. It follows by mathematical induction that $A_n A_{n-1} = A_1 A_0/n$ for $n = 1, 2, \ldots$. Since $0 \leq \sin x \leq 1$ in $[0,\pi]$, $A_{n+1} \leq A_n \leq A_{n-1}$ $(n \geq 1)$. Hence

$$\frac{A_1 A_0}{n+1} = A_{n+1}A_n \leq A_n^2 \leq A_n A_{n-1} = \frac{A_1 A_0}{n}$$

or $\sqrt{n/(n+1)}\sqrt{A_1 A_0} \leq \sqrt{n}A_n \leq \sqrt{A_1 A_0}$. Finally by the Squeeze Theorem, it follows that as $n \to \infty$, the desired limit $= \sqrt{A_1 A_0} = \sqrt{2\pi}$.

Problem 101. Nested Polygons

Starting with a unit circle, inscribe a square in it. Inscribe a second circle in the latter square, then a regular octagon in the second circle, etc., each time inscribing a circle in the last regular polygon and then inscribing a regular polygon of twice the number of sides of the previous one in the previous circle. Determine the limit of the resulting sequence of radii.

Michael Woltermann, Washington and Jefferson College, generalized the problem by starting with a regular p-gon instead of a square and obtained in a similar way as the previously published solution that the limit is $(p/2\pi)\sin(2\pi/p)$.

S-21. Simultaneous Equations

Determine all the solutions of the simultaneous equations: $x^5 + y^5 = 211(x+y)$, $x^5 - y^5 = 55(x-y)$.

Solutions by the proposer. Eliminating y, we obtain an equation of degree 25, so that there are 25 solutions in general. Factoring the equations we get

$$(x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4 - 211) = 0$$

and

$$(x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4 - 55) = 0.$$

This leads to four subsystems of equations:

(1) $x + y = 0$, $x - y = 0$,
(2) $x + y = 0$, $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 55$,
(3) $x - y = 0$, $x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 211$,
(4) $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 55$,
$x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 211$.

Note that these systems are symmetric in x and y so that if $(x,y) = (a,b)$ is a solution so is $(x,y) = (b,a)$. The solutions of (1), (2), and (3) follow easily and give 9 solutions. System (4) is solved by letting $x + y = s$ and $xy = p$ to give $s^4 - 5ps^2 + 5p^2 = 211$, $s^4 - 3ps^2 + p^2 = 55$. We can eliminate s, since by subtraction $s^2 = 2(p^2 - 39)/p^2$, giving $p^4 + 133p^2 = 60084$. Thus $p^2 = 36$ and -13^2. The 16 other solutions now follow.

Also by Edward T. H. Wang and incompletely (17 solutions) by Jiu Ding.

Editorial note. By Bezout's theorem the number of solutions of two polynomial equations is given in general by the product of their degrees.

S-22. Solutions in Terms of Radicals

Determine all real a, b, c, d, and e such that the equation

$$x^6 + ax^4 + bx^3 + cx^2 + dx + e = 0,$$

is solvable in terms of radicals.

Solution by E. M. Kaye, Vancouver, BC. Let r_1, r_2, r_3, r_4, r_5, r_6, be any numbers which can be expressed in terms of radicals and whose sum is 0. Then

$$a = \sum r_1r_2, \quad b = -\sum r_1r_2r_3, \quad c = \sum r_1r_2r_3r_4,$$
$$d = -\sum r_1r_2r_3r_4r_5, \quad \text{and} \quad e = r_1r_2r_3r_4r_5r_6.$$

Here the sums are symmetric over the r_i's. If these sums are given in simplified form, it is highly doubtful that one can recover the r_i's as radicals.

Also solved by the proposers.

Problem 108. A Non-Intersecting Line

Let S be the set of all points in a plane whose rectangular coordinates are (a,b) where a is a prime number and b is a square number $\neq 0$ and let C be the set of all circles centered at the points of S with radii 0.32. Prove that there exists a line $y = mx$ $(m > 0)$ which does not intersect any of the circles.

Solution by the proposer. We will show that the line $y = 6x$ does not intersect any circle centered at points of S of radius less than $2/\sqrt{37} = 0.32879\ldots$, from which the desired result follows. If x is a prime p, then $y = 6p$. We now show that $6p \neq t^2 - 1$, t^2, or $t^2 + 1$ for any integer t and from which it follows that $|6p - t^2| \geq 2$. First, $6p = t^2 - 1$ implies t is odd, so $t^2 - 1 \equiv 0 (\text{mod } 4)$, $p = 2$, but $6 \cdot 2 + 1 = 13$ which is not a square. Next, $6p = t^2$ implies $2 \mid p$ and $3 \mid p$ which is impossible. Finally, $6p = t^2 + 1$ is impossible since 5 is not a square mod 6. Now consider circles with centers $(p, 6p \pm 2)$ which are tangent to the line $y = 6x$. It follows that both circles have radii $= 2/\sqrt{37}$. Therefore if all the circles centered at S have radii $< 2/\sqrt{37}$, none of them will intersect the line $y = 6x$.

Problem 109. Maximum Problem

Determine the maximum value of

$$(x+y+z)(\sqrt{a^2-x^2}+\sqrt{b^2-y^2}+\sqrt{c^2-z^2})$$

where a, b, c are constants.

Solutions by Edward T. H. Wang, Wilfrid Laurier University. Let $k = a + b + c$ and $s = x + y + z$. We will show that if f denotes the given function, then $f \leq k^2/2$ with equality iff $x = a/\sqrt{2}$, $y = b/\sqrt{2}$, $z = c/\sqrt{2}$. It follows from Cauchy's Inequality that

$$\begin{aligned}(\sqrt{a^2-x^2}&+\sqrt{b^2-y^2}+\sqrt{c^2-z^2})^2 \\ &\leq (a-x+b-y+c-z)(a+x+b+y+c+z) \\ &= k^2 - s^2.\end{aligned}$$

Hence, $f^2 \leq s^2(k^2 - s^2) \leq k^2/4$ with equality iff

$$\frac{a+x}{a-x} = \frac{b+y}{b-y} = \frac{c+z}{c-z} \quad \text{and} \quad 2s^2 = k^2.$$

Generalization by the proposer. Let $F = \sum x_i \sum \sqrt[n]{a_i^n - x_i^n}$ where $n \geq 2$ and the sums here and subsequently are over $i = 1, 2, \ldots, m$. Letting $x_i^n = s_i^n \sin^2 \theta_i$,

$$\begin{aligned}F &= \sum a_i(\sin\theta_i)^{2/n} \cdot \sum a_i(\cos\theta_i)^{2/n} \\ &= \sum a_i^2(\tfrac{1}{2}\sin 2\theta_i)^{2/n} \\ &\quad + \sum a_i a_j[(\sin\theta_i\cos\theta_j)^{2/n} + (\sin\theta_i\cos\theta_j)^{2/n}]\end{aligned}$$

By the power mean inequality,

$$\begin{aligned}[(\sin\theta_i\cos\theta_i)^{2/n} + (\sin\theta_i\cos\theta_i)^{2/n}] &\leq 2\left[\frac{\sin(\theta_i+\theta_j)}{2}\right]^{2/n} \\ &\leq 2^{1-2n}.\end{aligned}$$

It now follows that

$$F \leq \sum a_i^2\left(\frac{1}{2}\right)^{2/n} + 2^{1-2/n}\sum a_i a_j$$

and with equality iff $\theta_i = \pi/4$ or $x_i = a_i/2^{1/n}$.

Also solved by Russell Euler, Jawad Sadek, Northwest Missouri State University; Brad Friedman, University of Illinois whose solutions were the same as the generalization for $n = 2$ and $m = 3$; D. Anderson, S. Smith, and Z. Wang.

Problem 112. A Definite Integral

Evaluate $\displaystyle\int_0^\infty \frac{\ln x\, dx}{(x^2+e^2)^2}$.

Equivalent solutions by Hongwei Chen, Christopher Newport University and Benjamin G. Klein, Davidson College. Starting with

$$I(a) = \int_0^\infty \frac{\ln x\, dx}{(x^2+a^2)},$$

we let $x = at$ to give $I = \pi(\ln a)/2a$ since

$$\int_0^\infty \frac{\ln t\, dt}{(t^2+1)} = \int_0^1 \frac{\ln t\, dt}{(t^2+1)} + \int_1^\infty \frac{\ln t\, dt}{(t^2+1)} = 0$$

by changing t to $1/t$ in the latter integral. Then differentiating I with respect to a, gives

$$-2a\int_0^\infty \frac{\ln x\, dx}{(x^2+a^2)^2} = \frac{\pi(1-\ln a)}{2a^2}.$$

Finally, setting $a = e$, the given integral $= 0$.

Also solved by A. Anderson, Parviz Khalili, S. Smith, and the proposer.

S-27. (Quickie) A Traveling Problem

Since the chauffeur saved half of 10 minutes on the outgoing trip, Mr. Smith walked for $65 - 5$ minutes. Since the chauffeur covered the distance Mr. Smith walked in 5 minutes, he drove at a speed of $(60)(4)/5 = 48$ miles per hour. Let T be the time in minutes for the car to make the trip from home to the railway station. Then, $(T - 5)48/60 = 4$ or $T = 10$.

S-28. (Quickie) A Cyclic Quadrilateral Inequality

Since $\angle B + \angle D = \pi$, at least one of these two angles is $\geq \pi/2$, say B. If M is the midpoint of arc AC, then $AB + BC \leq AM + MB = 2AM$. If N is the midpoint of AC, then

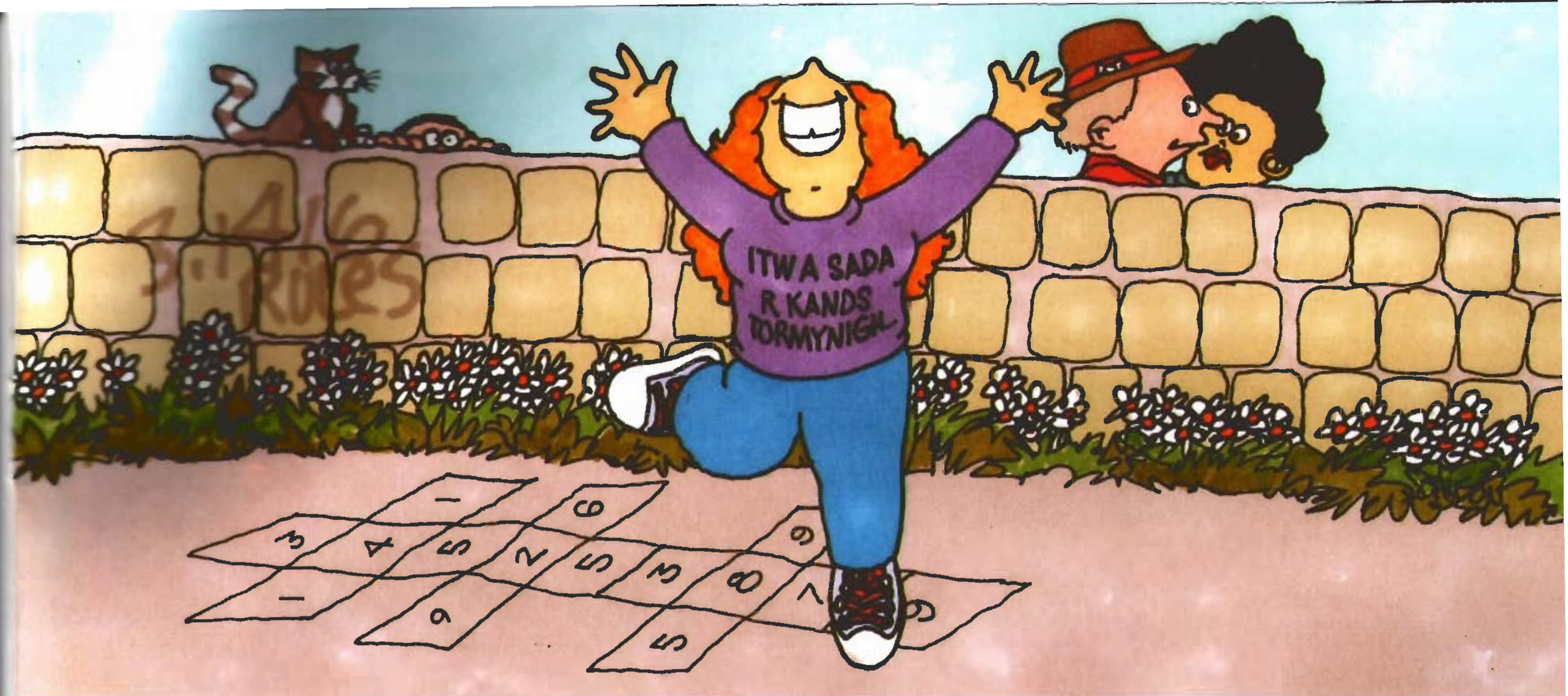
$$\sqrt{2}AC = 2\sqrt{2}AN = 2\sqrt{2}AM\sin B/2 \geq 2AM$$

(angles inscribed in the same segment are equal).

S-30. (Quickie) Three Covering Disks

Let A, B, C be the vertices of an inscribed equilateral triangle in the disk D of radius r. Then the smallest radii for the three covering disks $= AB/2 = \sqrt{3}/2$ and with centers at the midpoints of the sides. Note that since the biggest chord of a disk is a diameter, one disk of radius $< \sqrt{3}/2$ would cover less than $1/3$ of the circumference of D.

Late solutions. **S-15:** *David E. Manes, Jerome Meyers, Patricia Solano;* **S-18:** *James P. Coughlin, David E. Manes, Jerome Meyers;* **106:** *James P. Coughlin.*



Wow! I have a great technique to recall those fun, crazy numerals composing perhaps everyone's all-in-all favorite real number — Pi!

How many digits of π can you recite from memory? At this suggestion, the lucky among you will find a curiosity lodged in a remote corner of your brain, somewhere between Great-Aunt Tillie's birthday and your killer recipe for lemon bars. It's a sentence with the following property: write down the number of letters in each word, and you'll have written out the first few digits of π. Chances are, it's the elegant "How I need a drink, alcoholic in nature, after the heavy lectures involving quantum mechanics," which is the only widely-recognized pi-mnemonic in the English language. It probably isn't

Can I find a ratio, immutable, to descry?
There can exist constant qualities — perfect roundness
For me the delicate star called pi
Speaks only for the splendid cry of heaven's soundness.

I'm still waiting for that one to catch on.

The rhymes are a nice touch, no? Obviously, pi-mnemonics are fertile ground for strutting one's stuff — poetic, mathematical, or otherwise. So why aren't there more? Perhaps it's because of the humbling experience that mnemonicizing provides. Forget your poetic turns of phrase, your rich idioms, your efforts to capture a realistic flow of dialogue. Unless you're incredibly lucky, none of these devices is going to mesh with π's tyrannical sequence of digits. Instead of gaining momentum with each word you write, you'll find that you face more and more restrictions as you go on. Not only do you need a six-letter word, you need a six-letter noun.

Then there are those who claim that this whole exercise is a little goofy. The relationship between π, the quantity, and the digits in its base-10 representation is nothing more than convention. The number of letters in a given English word has nothing to do with the word's meaning. The job of the mnemonic-er is to find a connection not between a meaning and a number, but between something incidental associated with a meaning and something incidental associated with a number. These people dismiss the exercise as trivial, or even artificial. Not I! What better way to impress your friends than with 1) your ability to rattle off an absurd number of decimal places of π, 2) your mastery of linguistic tricks, and 3) your ability to write a cute story or snappy poem?

As it happens, you've wedded yourself to some pretty weird problems by your choice of digits and dialect. In the first four digits of π, the number one appears twice. The obvious thing to do is to cast your mnemonic in the first person so you can use the word "I." After all, your only other obvious choice is "a." But are pi-mnemonics to be relegated to the realm of the narrative? Surely we can do better. Another issue: can you write a past-tense mnemonic? Of course there must be a way, but the short word-lengths required by the first few digits prohibit simply tacking on -ed suffixes to most words. On the surface, it looks as if most mnemonics will be declarations, imperatives, or fragmented exclamations. Or will they?

The proverbial astute reader will by now have guessed the challenge. In an effort to combat the shortage of pi-mnemonics known to the English language, *Math Horizons* asks that you conceive, create, and convey your own mnemonics. Poetry and narratives are especially encouraged, but any mnemonic devices will do. And one shouldn't feel limited to pi-etry. Perhaps there's more music to be uncovered in e or root 5. Whatever solution to the challenge your creative Muse whispers in your ear, send it to: Deanna Haunsperger, Math Department, Carleton College, Northfield MN 55057, or e-mail `dhaunspe@carleton.edu` by May 15, 1999.

As a reward for your mnemonicizing, you could see your entry immortalized in a future issue of Math Horizons. You might even get a free copy of *Lion Hunting and Other Mathematical Pursuits*, by Ralph P. Boas, Jr. At any rate, you can be sure that you'll never forget the tenth decimal place of π again. Beware, Aunt Tillie! You may not get a birthday present this year.

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