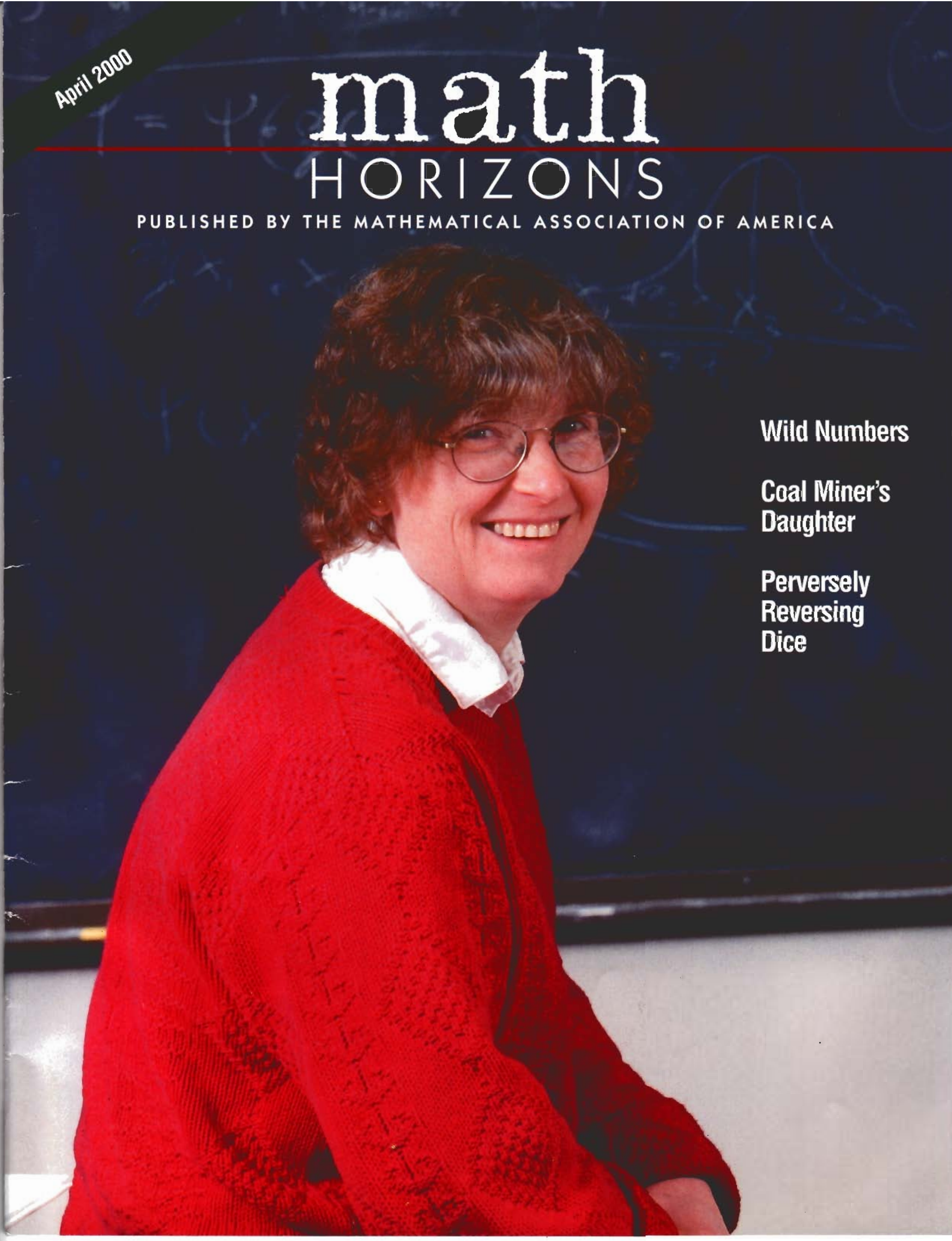
April 2000

math HORIZONS

PUBLISHED BY THE MATHEMATICAL ASSOCIATION OF AMERICA



Wild Numbers

Coal Miner's Daughter

Perversely Reversing Dice

Rhapsody in White

At the recent Breckenridge (Colorado) International Snow Sculpture Championships a Macalester College team, under the direction of sculptor Robert Longhurst of Chestertown, New York, took a silver medal, as well as two other awards. The team consisted of Longhurst, Macalester mathematics professors Dan Schwalbe and Stan Wagon, Macalester mathematics major Andy Cantrell, and John Bruning (Rochester, New York; non-sculpting team manager). The elite international snow-sculpting event had 17 teams from Russia, England, Switzerland, Finland, The Netherlands, Mexico, Canada, and the U.S. The Minnesota team was sponsored by Wolfram Research, Inc., the developers of *Mathematica*.

The team started with a 12-foot high block of compacted snow and carved it into an Enneper surface, truncated to eliminate the self-intersections and maximize the overhang and dramatic impact. The swooping boundary curves and the rhythm and symmetry of the piece led to the title: *Rhapsody in White*. The Enneper surface is an example of a minimal surface: every point is a saddle point, which gives it great structural strength, thus allowing the carving to be thin and to have a substantial overhang.

In addition to taking second place, the team was awarded the Artists' Choice award (voting by the other teams) and the People's Choice award (voting by the approximately 10000 people who view the pieces on the final weekend).

Longhurst had carved wooden models of the surface, so he had a detailed sculpting plan in mind. The first step was to make a 6-lobed rose prism, which was done with the aid of a plastic template and an ice-fishing drill to remove the dense snow. Then it was a matter of carefully removing everything that shouldn't be there. The last step, removing the supporting plugs holding the overhang up, was done on live TV on the final morning. The overhang held its shape for 12 days following the event. Pictures from the event, including shots of the other sculptures, may be seen at `www.math.macalester.edu/snow2000`. Wagon, in his acceptance speech said:

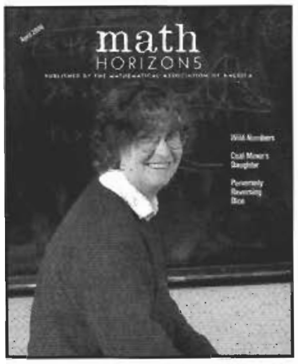
"Julia Child has said: 'Il faut mettre les mains dans le pâte.' If one wants to be a baker, one must put one's hands in the dough. Four members of our team are mathematicians and we spend a lot of time looking at images on a computer screen. But, both for us and for the viewers of our work, true understanding can be obtained only by interacting with the piece in a truly 3-dimensional way. This is what snow allows us to do. In a very short period of time and with a minimum of tools we can sculpt a complicated shape and so learn much more about it. It's a glorious opportunity and tremendous fun. Thank you all for creating an environment in which we can accomplish our goals."

Andy Cantrell and Robert Longhurst

Photos courtesy of Stan Wagon.

Finished, front view

Finished, 3/4 view



Math Horizons is for undergraduates and others who are interested in mathematics. Its purpose is to expand both the career and intellectual horizons of students.

Math Horizons (ISSN 1072-4117) is published four times a year; September, November, February, and April by the Mathematical Association of America, 1529 Eighteenth Street, NW, Washington, DC 20036. April 2000 Volume VII, Issue 4. Periodicals postage paid at Washington, DC and additional mailing offices. Annual subscription rates are $25.00 for MAA members, $35.00 for nonmembers, and $40.00 for libraries. Bulk subscriptions sent to a single address are encouraged. The minimum order is 20 copies ($130.00); additional subscriptions may be ordered in units of 10 for $65.00 per unit. For advertising rates or editorial inquiries call (202) 387-5200. Printed in the United States of America. **POSTMASTER: Send address changes to *Math Horizons*, MAA Service Center, PO Box 91112, Washington, DC 20090-1112.**

THE MATHEMATICAL
ASSOCIATION OF AMERICA
1529 Eighteenth Street, NW
Washington, DC 20036

In this issue

Cover photo by Martin Griff.

Instructions for Authors

Math Horizons is intended primarily for undergraduates interested in mathematics. Our purpose is to introduce students to the world of mathematics outside the classroom. Thus, while we especially value and desire to publish high quality exposition of beautiful mathematics we also wish to publish lively articles about the culture of mathematics. We interpret this quite broadly—we welcome stories of mathematical people, the history of an idea or circle of ideas, applications, fiction, folklore, traditions, institutions, humor, puzzles, games, book reviews, student math club activities, and career opportunities and advice. We welcome electronic submission of manuscripts. Please send to `skennedy@carleton.edu`. If submitting by mail, please send two copies to Steve Kennedy, Department of Mathematics and Computer Science, Carleton College, Northfield, MN 55057.

Subscription Inquiries

e-mail: maaservice@maa.org **Web:** www.maa.org/pubs/mh.html **Fax:** (301) 206-9789
Call: (800) 331-1622 or (301) 617-7800 **Write:** Math Horizons, MAA Service Center, P. O. Box 91112, Washington, DC 20090-1112

DEANNA HAUNSPERGER and STEPHEN KENNEDY

Coal Miner's Daughter

The first thing that strikes you is that she's charming. Only later do you realize that she is probably the smartest person you've ever met. Somehow, you don't expect mathematical genius and a charming, genuinely warm personality to coexist. Yet, here sitting down to lunch with us is an existence proof: Professor Ingrid Daubechies of Princeton University, MacArthur Fellow, member of the National Academy of Sciences, the mother of wavelets. Fuzzy-haired, bespectacled, quick to smile and a world-class storyteller, she's telling us about growing up in a coal-mining town in Belgium in the sixties.

"I grew up in a very small town. My father worked in a coal mine. He was an engineer in a coal mine, and of course while I was growing up I didn't notice, but in retrospect coal-mining towns are very special kinds of small towns. I mean, there is just one big employer, and he controls the whole life, even the social life, of the town. My mother, well, for her generation in Belgium it was not common to have university education, but she did. She had expected to have a career, but after marrying my father she didn't. Partly because there was no opportunity, but also because in this very small paternalistic town, two generations behind the wide world, it just was not done for wives of engineers to work. There was one wife who worked, she was a nurse, and everybody knew that that was why her husband never got a promotion. It wasn't said that way, of course, and even then they couldn't write things down that way. So my mother didn't work.

"I remember when we were little she did a lot with us kids. As we grew up we became more independent because she wanted to give us our independence, but she was also very bitter at not having any bigger framework for her own life. She went back to college when I went to university. We had moved by then. She went for a different degree because her first degree, in economics, had become obsolete — she hadn't worked for twenty years. She got a second degree in criminology and worked for about twenty years as a social worker. She worked with troubled youths, trying to monitor them, and to help them, and to give them decent lives.

"She had met my father while they were students at different universities at a meeting that brought together students from universities in Belgium. She met him, and later they decided to get married. They married in 1952, and he started work at the coal mine. At this coal mine they would only hire engineers who were either married or engaged to be married very soon. They did not want trouble with single men around.

"My father really would have liked, himself, to become a scientist, to become a physicist. He was really mostly interested in physics, but he became a mining engineer. His parents were very poor. They came from a coal-mining region, and for them an educated person was a coal-mining engineer. They had never seen any other profession. Also, there was a very good engineering school in that area, so they made a deal with him when he was growing up that they would not, like their friends, save for their retirement. My grandfather actually didn't work at the coal mine, he worked in a glass factory, and they lived in a house that was owned by his employer. In these company towns people didn't live in houses that belonged to them, so everybody would save so that they could buy a small house or a small apartment to live in after their retirement. But my grandparents made a deal with my father that they wouldn't do that; instead they would pay for his education and

DEANNA HAUNSPERGER and STEPHEN KENNEDY are the editors of *Math Horizons*. Their conversation with Professor Daubechies took place in January 1999.

then he would take care of them when they retired. So, that's how it happened that he became a coal-mining engineer — because it had all been planned that way, and he only discovered while at the university that there were other choices. He was at a school which was an engineering school. He wouldn't have been able to explain a change to his parents; they would have been so worried. As a coal-mining engineer they knew he would be able to support them.

"The region where my father's parents were born and lived was a very poor region. The reason my grandfather didn't work in the coal mines was that his mother really wanted one of her children not to go down the coal mine because that was the time when you would die young if you went into the coal mines. They didn't know how to prevent black lung disease; everybody died young. The coal mines were just disputing the fact that it was anything to do with work in the mines, so you didn't get any compensation either. My grandfather had been sickly when he was little and he was the first-born, so his mother wanted him to work elsewhere. Generations in my family are very long, so I'm talking the end of the 19th century when my grandfather was born. He left school when he was nine. I said this was a poor region, and this was before child-labor laws. His mother, my great-grandmother, had arranged a job for him in a big glass factory; he was in packing. But then, these things are incredible, by the time he was 14, he had in various accidents lost a finger and an eye in the glass factory. I think of my children, and I think — how is this possible? My father always says of his father that he was really very smart. He went to evening school at some point. He was in packing all of his working life, and he became a foreman. At some point, for a very complicated delivery, they had to make a case the inside of which was the intersection of two cylinders, but he had to make it in wood, of course, and fold it out of plywood. He had tried with ellipses and somehow it never fit, so he actually went back to evening school to study mathematics in order to learn how to do that. Another foreman had explained how to do it, but he wanted to know what it looked like. If you took a cylinder and then unfolded it, what would it look like? How do you actually compute that? So, he went back to school."

An Education in Physics

Ingrid attended the Dutch Free University in Brussels where she studied physics. She held a research position in physics at that same university until 1987 when she came to the United States. Today, though, she asserts, "I'm a mathematician."

"My father was always interested in mathematics, and he was always interested in explaining things to me, and I liked it. I would ask questions. I would usually get answers which were much longer than I hoped for, so I am trying with my children to yes, give answers,

but maybe not go beyond so far. I remember liking mathematics when I was little, but I actually did major in physics. I think I majored in physics because it was my father's dream to become a physicist; he explained to me things about physics. He went to extra Open University courses whenever he could. Sometimes they would organize a series of physics lectures, and he would go.

"Physics just seemed to be a very noble choice because of my father's influence. It was something intermediate between what I really wanted — mathematics — and what my mother really wanted — which was that I would have become an engineer. She was a bit worried about all this science. She thought scientists were like artists, they really cannot make a good living. An engineer can always find a good job.

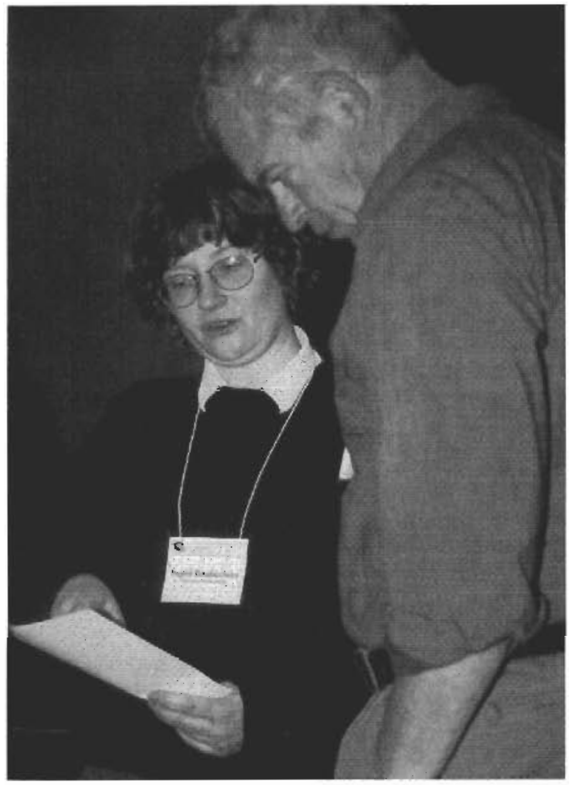
"With a free choice, I think I probably would have chosen mathematics. I don't know. I liked physics very much. I especially liked some physics classes that we had. At some point I was considering switching between math and physics, and I decided to stay in physics because of one particular course which I thought was wonderful, in which we were going beyond geometric optics. If you go to the Kirchoff-Fresnel theory of optics, you actually see, and can compute, that a lens, in fact, computes a Fourier transform, which I think is wonderful. I think this is mind-boggling, that a lens would, in fact, compute a Fourier transform and this is used in some optical computing. This was marvelous. In fact, this course was really a course in applied mathematics. It was labeled as a physics course, and it was wonderful, and so I stayed with physics. I don't regret it. As a result I have learned a whole lot of things that I wouldn't have learned in a standard math curriculum. And the math that I wanted and needed I have learned by myself anyway.

"I think I think like a mathematician; I switched from theoretical particle physics to more mathematical physics because I felt that people who were really good at particle physics had an intuition about which I had no clue. I felt like I could learn how to read those papers, but it was like learning a language without understanding the meaning of the words, which I didn't like at all. It's hard to describe how I think. Even in analysis, I don't think in formulas. Although when I work something out, I *do* compute a lot. I have some kind of mechanical or geometrical way of thinking, I don't really know where that comes from.

"Anyway, in Belgium, undergraduate education is really different from here in that you track very, very early on. When you register for the university, you have to say what you're going to major in. So you get very few courses outside your major or outside things related to your major. For physics, you get a lot of math, you get some chemistry, but you don't get any liberal arts courses. I think you could go sit as an auditor in some of these courses, but really there's no time; you don't choose your own courses. You say 'I will major in physics,' and then the courses are specified except that in later years, you have some choices, you get to choose one of four. In the first two years everything is completely chosen for you, and it's quite a heavy schedule. It's a heavier schedule than I see here, but the result is that you can do much less independent work. I think a schedule where you put together a combination yourself and where you're encouraged to do a lot of independent work is actually better.

"I was tracked with physics, so I had a lot of math courses, especially the first two years. And when I had majored, I had seen a lot of physics courses that would be at graduate level in the States because you cannot cram four years full of physics courses and not get to that level. Things are not organized so much by semester as they are for a whole year, so many courses were a full year. In the third year, that was really the heaviest year, we had 13 different physics courses, and we had 5 weeks of labs. Lots of that physics I have forgotten.

"At the graduate school level, however, in most universities in Belgium, you don't get courses any more. There is a movement there now to get some what-they-call third-cycle courses which are graduate courses, but mostly you're left to learn extra things on your own or with your advisor. You're assigned or find a research topic; right away when you arrive you start working on papers. You also have a teaching schedule, and you're expected to have your PhD in 5 or 6 years. I never had a teaching schedule because I had a special fellowship, but on a teaching assistantship, you would teach (be in the classroom) 30 hours a week. I would typically have 8–10 hours a week. The things you would be teaching would be problem sessions. Of course, you were there for the students, and typically the only person there for the students to ask questions from because they would not dare to approach a professor. You don't have control over what you're doing; you'd be given a list of problems, and you'd answer questions about them. It was very frustrating because it meant you had zero input into what you did. That persists even after the PhD, for a while."



Daubechies discussing mathematics with R. Gundy (Rutgers)

A New Paradigm

Wavelets are everywhere these days. Wavelets are a new method for encoding and compressing information (see box). They are being used in image compression (the FBI's files of approximately 200 million fingerprints are being converted to wavelet-compressed electronic images), also sound and video compression, medical imaging, and geological exploration. Ronald Coifman of Yale University used wavelet techniques to remove the noise from a century-old recording of Brahms playing one of his own compositions.

The most exciting thing about wavelets might just be the way that they are drawing together people and ideas from so many different fields of science: mathematicians, physicists, geologists, statisticians, computer scientists, engineers of all kinds. In fact the history of the idea has roots in all of these fields and more. Yves Meyer has identified precursors to the idea in mathematics, computer science, image processing, numerical analysis, signal processing, studies of human and computer vision, and quantum field theory. The short version of the history has geophysicist Jean Morlet and mathematical physicist Alexander Grossmann introducing the idea in the early 1980s. Meyer and Stephane Mallat pieced together a mathematical framework for wavelets in the mid-eighties. In 1987 Daubechies made her famous contribution of a family of wavelets that are smooth, orthogonal, and equal to zero outside a finite interval. Thus, in a stroke, accomplishing what everyone supposed impossible and making wavelets very much more applicable.

"How did I get started on wavelets? For my PhD work, I had worked on something in quantum mechanics which are called coherent states. This is a tool to understand the correspondence between quantum mechanics and classical mechanics. So you try to build functions that are well-localized, that live in Hilbert space, but that correspond as closely as you can with being in one position, in one momentum in classical mechanics. I had worked in Marseilles with Alex Grossmann, and Alex was one of the people who really started the whole wavelet synthesis. There are roots in pure mathematics, in many different fields, but the synthesis really, I feel, was created by Alex Grossmann and

ED ABOUFADEL

A Short Primer on Wavelets

Wavelets have many applications, including the processing of fingerprint images by the FBI, the analysis of earthquake data, and the development of a text-to-speech system. As a window into understanding wavelets, consider the problem of approximating complicated functions with simpler functions. In calculus, we learn how to approximate functions with Maclaurin and Taylor polynomials. For example, if f is a function defined near $t = 0$ then we can approximate it near $t = 0$ with a Maclaurin polynomial such as:

$$f(t) \approx f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3.$$

In Figure 1, we see how $\ln(t+1)$ can be approximated by $0 + 1t - t^2/2 + 2t^3/6$

When using Maclaurin polynomials, we call the functions $\{1, t, t^2, t^3, \dots\}$ *basis functions*, and our approximation is created by adding multiples of the basis functions together.

A more powerful version of function approximation is *Fourier analysis*. In Fourier analysis, sine and cosine functions are used for the basis functions, instead of polynomials, and we attempt to have a good approximation on a fixed interval such as $[0, 2\pi]$. The goal is to decompose a function by thinking of it as a combination

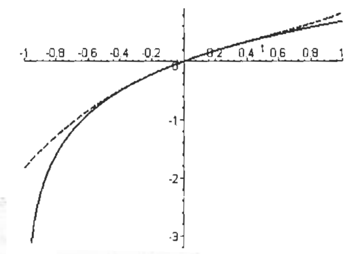


Figure 1. *Approximating* $\ln(t+1)$ *with a Maclaurin polynomial*

of trigonometric functions with different *frequencies*. For example, we can approximate $\ln(t+1)$ with the following Fourier series:

$$\ln(t+1) \approx 1.301 - .155\cos(t) - .053\cos(2t) - .026\cos(3t) - .528\sin(t) - .294\sin(2t) - .202\sin(3t)$$

For a graph of $\ln(t+1)$ and the Fourier approximation, see Figure 2.

This approach gives a good approximation in general, although the error is worse near $t = 0$ and $t = 2\pi$. The results are even worse for jagged functions, such as the characteristic function on the time interval $[0, \pi]$. (This is the function that is equal to 1 on the interval $[0, \pi]$ and 0 everywhere else.) In Figure 3, this function is approximated by a sum of sines and cosines.

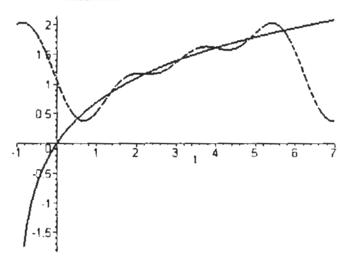


Figure 2. *Approximating* $\ln(t+1)$ *with a Fourier series*

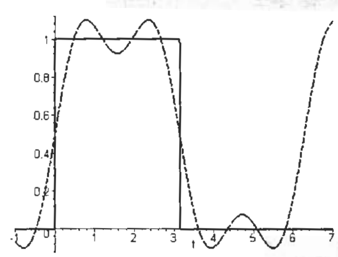


Figure 3. *Another example from Fourier analysis*

ED ABOUFADEL is an Associate Professor of Mathematics at Grand Valley State University and the author, with Steven Schlicker, of *Discovering Wavelets*, Wiley, 1999. For more information on wavelets, begin at the *Discovering Wavelets* web site: `www.gvsu.edu/mathstat/wavelets.htm`

Jean Morlet. Morlet was not happy with what was called the wavelet transform, and he wanted something else, so he invented an algorithm. But he had no mathematical theory behind it. Alex Grossmann realized there was an analogy with the coherent state formalism in quantum mechanics. He recast it in terms of a group representation and then people started saying 'But we've been doing that all along in a different context.' And they were right; they had been doing it all along. So that's what made the synthesis happen that made the jump from mathematicians to engineers.

"I knew Alex Grossmann very well from my thesis work, and I was looking for something else to start working on. This was a time when I had many changes in my life. Before my husband, I had a long relationship, and I had just left him, and I was looking for something else, for changes. So I changed research topics. I started working on wavelets in '85. I started working on wavelets and I met Robert [Calderbank, the mathematician to whom she is married] within a period of six weeks in '85.

"At this time geologists were doing windowed Fourier transforms, so they would window [look at a short time segment of a signal], then Fourier transform [approximate that bit of signal with combinations of sines and cosines]. This means that for very high frequency things that live in very short time intervals — these are called transients — if you've determined your window to be this wide, then you need a whole lot of high frequency functions to capture that behavior. So, you could of course make your window very narrow, but then you don't capture a lot of them. Morlet didn't like that aspect of the windowed Fourier transforms, so he said 'After all, I'm using location and modulation, let's do it differently. Let's take one of these functions that have some oscillations and let's put that in different places and squish it so that I have a different thing,' really wavelets. Now he didn't really formulate this precisely. Actually the name comes from there because in geology when you have different windows, which determine the shape of these multi-layered functions, they call them wavelets. So he called his transform 'a transform using wavelets of constant shape' because the other ones didn't have constant shape. If you adjust the window, or you modulate this way, then of course they look different. In his case, the wavelets look the same, they were just dilated versions. He called them wavelets of constant shape, but then once they left that field, there was no other thing around called wavelets, so he just dropped the 'of constant shape' to the great annoyance of geophysicists because in their field it has another meaning.

"Wavelets were not really something that were a trend in geophysics, Morlet

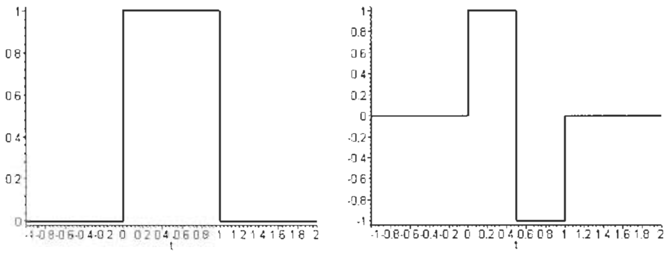


Figure 4. *The "father" and "mother" Haar wavelets*

Wavelet analysis is designed to better handle this type of function, because of the focus on the *time intervals* where functions are defined. In the example above, $[0, \pi]$ is an important time interval, so wavelet basis functions are needed that will emphasize this interval in the decomposition. An easy-to-understand set of basis functions that have this property are the *Haar wavelets*. The "father" and "mother" Haar wavelets can be found in Figure 4.

The Haar wavelets have properties which are different than sine and cosine. For instance, these wavelets are not periodic. They are zero for most of the real line $(-\infty, \infty)$.

There is also a way of creating these functions that is different. The basis functions for Fourier analysis come from *scaling* the sine and cosine function (in other words, we start with $\sin x$, and $\cos x$ and then scale these functions by creating $\sin 2x$, $\cos 2x$, $\sin 3x$, $\cos 3x$, ...).

The basis functions for wavelet analysis come from *dyadic scaling*, which means that the scaling coefficients are only powers of 2. If Fourier analysis was done with dyadic scaling, then we would only use functions like $\sin 2x$, $\cos 2x$, $\sin 4x$, $\cos 4x$, $\sin 8x$, $\cos 8x$, etc. We also use *translating*, so that we can slide the scaling function to any important time interval. The function that is scaled and translated is called, not surprisingly, the *scaling function*, or "father" wavelet. We use ϕ to stand for the scaling function, and some of the "children" of $\phi(t)$ are $\phi(2t)$, $\phi(2t-1)$, $\phi(4t)$, $\phi(4t-3)$ and $\phi(8t)$.

Other wavelets are created by combining these wavelets. For example, the "mother" wavelet $\varphi(t) = \phi(2t) - \phi(2t-1)$.

In Figure 5, we see how $\ln(t+1)$ can be approximated on the interval $[0, 1]$ by a series of Haar wavelets.

During the 1980s, Ingrid Daubechies developed a special type of scaling function ϕ that had three properties. First, the function is equal to zero outside of the interval $[0, 3]$. Second, for any two different integers k and l

$$\int \phi(t-k)\phi(t-l)\,dt = 0$$

This condition is called the *orthogonality condition*. Third, you can approximate constant and linear functions with no error, which is actually quite remarkable.

Combining these requirements, Daubechies deduced the following identity:

$$\phi(t) = \frac{1+\sqrt{3}}{4}\phi(2t) + \frac{3+\sqrt{3}}{4}\phi(2t-1) + \frac{3-\sqrt{3}}{4}\phi(2t-2) + \frac{1-\sqrt{3}}{4}\phi(2t-3)$$

From this identity, you can generate Daubechies' scaling function, which is pictured in Figure 6.

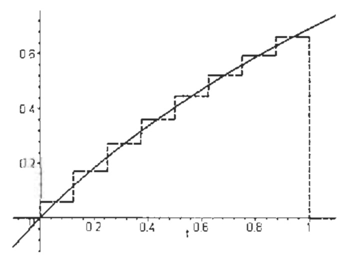


Figure 5. *Approximating* $\ln(t+1)$ *with a wavelet series.*

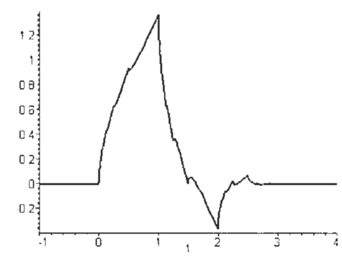


Figure 6. *Daubechies' scaling function.*

just came up with it. In harmonic analysis, people had been looking at, not exactly the same way, but something similar for ages. It goes back to Littlewood-Paley theory [circa 1930], and even the integral transform formula that Grossmann and Morlet wrote, because Morlet had no real formulas, is a transform that you find in Calderón's work in the sixties. So in some sense they had reinvented the wheel. In another very real sense they had looked at it completely differently. For Calderón it was a tool to carve up space into different pieces on which he would then use different techniques for estimates. Grossmann and Morlet gave these wavelets some kind of physical meaning, in a certain sense, viewing them as elementary building blocks which was a different way of looking at it. And Yves Meyer later told me that when he read those first papers by Grossmann, it was very hard for him because it was a different style. It took a while before he realized in what sense it was really different, because at first you see the formulas there, and you say, 'well, yeah, we've been doing that for 20 years,' but then you realize that here was a different way of looking at it: a new paradigm shift."

A Link in a Chain

To most mathematicians it appears that wavelets sprang full-grown from the foreheads of Grossmann, Morlet, and Daubechies and then were immediately grabbed by engineers and scientists. It is unusual for a piece of mathematics to find so many applications so quickly. Daubechies with her ability to talk the languages of physics and engineering and mathematics is, to a large degree, responsible for the building of so many bridges between the groups.

"In pure mathematics the idea was developed starting in the thirties, then in greater detail in the sixties. It was a very powerful tool which lived in a relatively small community in mathematics, and outside the small community, I felt it spread rather slowly. For example, in quantum mechanics I think some of these techniques would have been useful to mathematical physicists earlier than they penetrated. I think it's because through Grossmann and Morlet there were intermediate people. I treasure every single electrical engineer I meet and with whom I can talk. I'm interested in talking with them, I think many mathematicians aren't, but I am. Even so, I find it hard to talk with many of them because we've been trained in completely different ways and the words mean different things. But I have found some I can talk to and I think it's very valuable when they are also interested in talking with me. I think it's easier for me because of this physics background I have

Continued on p. 28.

ALLEN J. SCHWENK

Beware of Geeks Bearing Grifts

What is a grift? It is circus slang for a swindle, a game rigged so the customer is at a disadvantage. The guys operating the games in the midway are known as grifters. As for geeks, I don't know how to define them, but I recognize them when I see them. If you are wondering what this has to do with mathematics, read on.

Three Nontransitive Dice

Let's play a game. We are given three nonstandard dice. One die is amber in color, the second is blue, and the third is crimson. Each die has six sides with six numbers, but these numbers are not the traditional 1, 2, 3, 4, 5, 6. Instead we see:

amber = A = {2, 2, 2, 11, 11, 14}
blue = B = {0, 3, 3, 12, 12, 12}
crimson = C = {1, 1, 1, 13, 13, 13}.

Now we are each to select a die and roll it once. Whoever rolls the higher number will win our simple game. Always the gentleman, I offer you first choice, so if you believe that one of the dice is superior to the other two, you certainly can select it and I will have to accept one of the two remaining. So which die do you select? You may think, "I'll take the one with the highest average number. That ought to be best in the long run." Plausible reasoning, but unfortunately you quickly confirm that each die will produce an average roll of 7. So perhaps they are equally good. Pick any one and let's see what happens.

Perhaps you decide to take A. If so, I will select B. Now there are 36 pairs that we can roll. Brute force listing shows that I will win 21 times and you will win 15. That is a 58.33% probability in my favor. Not overwhelming, but Las Vegas prospers on a smaller edge than this. We say die B dominates die A, or B → A.

OK, so now you realize A was not the best choice. You ask for B. If so, I now choose C. Again listing all 36 pairs, we find that I win 21 times and you win 15. Die C dominates B, or C → B.

Now you see your best strategy, you ask for C. Fine, I will take A. The listing shows ... 21 wins for me and 15 for you! Die A dominates C.

What we have just discovered is that A → C → B → A. Domination is a nontransitive relation on this set of dice. And if a geek bearing dice offers you first choice, beware!

The concept of nontransitive dice is not new, although the triple shown here is. Martin Gardner [1] presented several sets of four nontransitive dice credited to Bradley Efron. One of these sets gives the second player a two-to-one advantage. Gardner [2] continued the discussion of nontransitivity in games and bets where we might expect transitivity. Tenney and Foster [3] generalized to construct sets of d dice with s sides for numerous pairs (d,s) with d and s at least 3.

Let's play again. Being a good sport, I will select first, and take A. I suppose you would like to have B, correct? And just for fun, let's roll twice each and take our total, the higher total wins. Recall that you have a 58.33% chance of beating me on the first roll, and also a 58.33% chance of beating me on the second roll. When we find our totals, is your probability of winning greater than 58.33%, equal to 58.33%, or less than 58.33%? Pause a moment and try to predict the answer. Does extending the game to two rolls enhance your expectation, leave it unchanged, or reduce it? Computing these probabilities is a lot more tedious. There are now 36 ordered pairs for each of us, so together we have $6^4 = 1296$ possible outcomes. Curiously, none are ties. We find 675 wins for A and 621 wins for B! That's right, not only has your winning edge been reduced, but it has been reversed to a winning margin of 52.08% for me! I call this a perverse reversal. In a single roll, B → A, but for a pair of rolls, A → B! To keep the conditions clear, we'll write the latter as 2A → 2B.

What happens when B opposes C? We have an even stronger reversal, 2B → 2C by a margin of 53.47%! And A → C also reverses to 2C → 2A by the same margin of 53.47%. For two rolls, our set of three dice is still nontransitive, but the dominating cycle has been reversed to

$$2C \to 2A \to 2B \to 2C$$

Perverse reversal, indeed.

After that surprise, are you ready to predict what will happen with three rolls? Here ties are possible, so we shall take the following point of view. If both players produce the same total for three rolls, we simply start over from scratch. They roll three more times, and either produce a winner or have another equal total leading to yet another repetition. In this way the game never ends in a tie. There are $6^6 = 46656$ possible outcomes. Let's say that t produce ties, a are wins for

ALLEN J. SCHWENK is Professor of Mathematics at Western Michigan University.

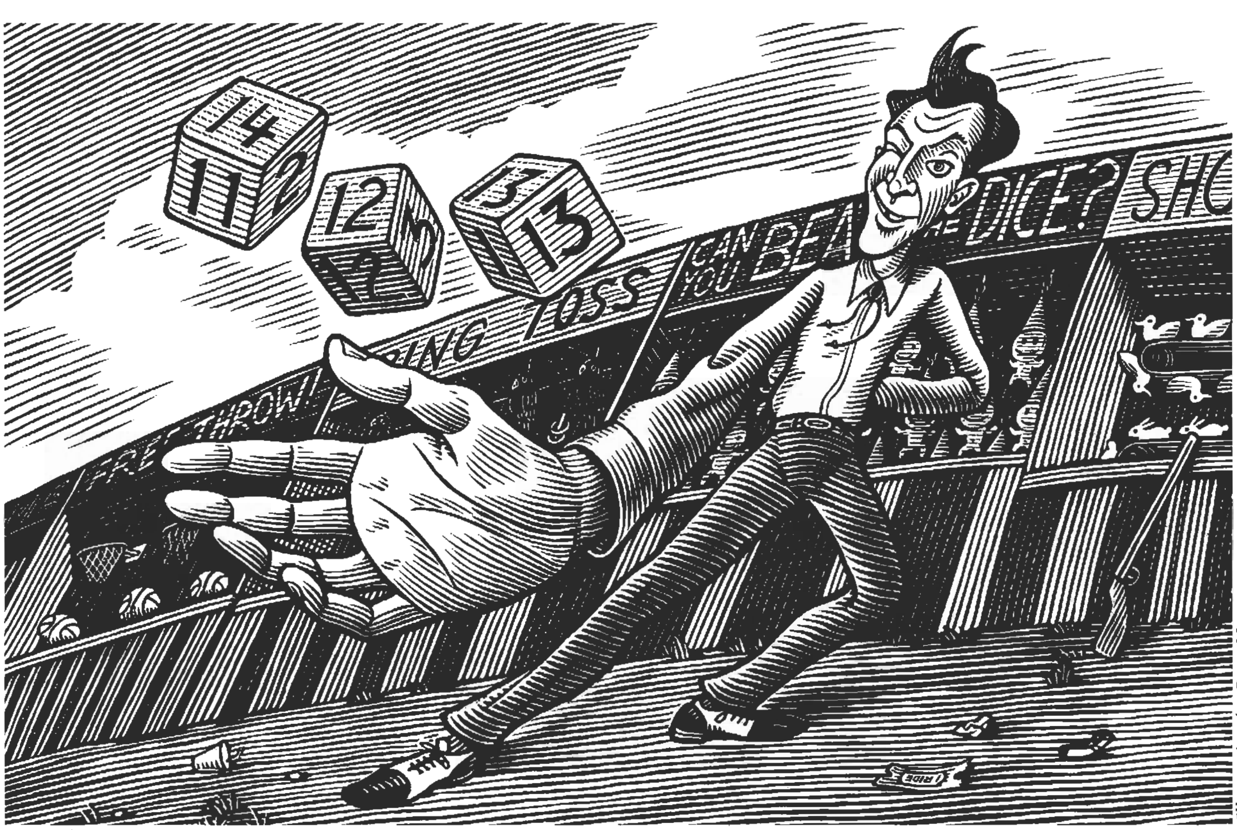


Illustration by Greg Nemec.

A and b are wins for B. Evidently $46656 = a + b + t$. Our tie-breaking rule effectively removes the t tying cases, giving A the probability $a/(a + b)$. In the case of three rolls, we find $a = 19818$, $b = 20358$, $t = 6480$. These perverse dice have tipped the scales back in the original direction. We find $3B \to 3A$ by 50.67%. And also $3A \to 3C \to 3B$, both dominating by the margin of 50.28%. Another nontransitive perverse reversal.

Several questions come to mind. For r rolls, will we always have a nontransitive triple? Which values of r will give the original order $rA \to rC \to rB \to rA$? And which will perversely reverse to $rA \to rB \to rC \to rA$? So far we have the original order for $r = 1$ and 3 and the reversed order for $r = 2$. It may seem that the margins are quickly dying out, but the first three cases may be misleading.

Finding the winning probabilities for r rolls is not as hard as it may first appear. For the amber die, we have sides of 2, 2, 2, 11, 11, 14. We represent this by a polynomial

$$a(x) = 3x^2 + 2x^{11} + x^{14}.$$

Similarly, the blue die gives

$$b(x) = 1 + 2x^3 + 3x^{12},$$

and the crimson one has

$$c(x) = 3x + 3x^{13}.$$

When A faces B in a single roll, B's winning margin is seen by observing that the term of x^{14} beats all six terms in $b(x)$. The two terms x^{11} each win three times, and the three terms x^2 each win one time. That's $1 \times 6 + 2 \times 3 + 3 \times 1$. Thus A has 15 winning combinations, similarly, B wins $3 \times 5 + 2 \times 3 = 21$ times. For two rolls we use

$$a^2(x) = 9x^4 + 12x^{13} + 6x^{16} + 4x^{22} + 4x^{25} + x^{28}$$

and

$$b^2(x) = 1 + 4x^3 + 4x^6 + 6x^{12} + 12x^{15} + 9x^{24}.$$

The winning pairs for A count up to be

$$1 \times 36 + 4 \times 36 + 4 \times 27 + 6 \times 27 + 12 \times 15 + 9 \times 5 = 675,$$

and the winners for B give

$$9 \times 31 + 12 \times 21 + 6 \times 9 + 4 \times 9 + 4 \times 0 + 1 \times 0 = 621.$$

By using a computer algebra system such as *Maple*, we can analyze r rolls quickly by computing the polynomials $a^r(x)$, $b^r(x)$, and $c^r(x)$. Of course it is still a chore to carry out the term-by-term comparisons.

However, using *Matlab* permits a matrix-vector approach that is even more convenient. The three polynomials are represented by column vectors whose $(i + 1)$st coordinate stands for the coefficient of x^i. Thus we have three vectors

$$\mathbf{a} = [0\ 0\ 3\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 2\ 0\ 0\ 1]^T$$
$$\mathbf{b} = [1\ 0\ 0\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 3\ 0\ 0]^T$$
$$\mathbf{c} = [0\ 3\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 3\ 0]^T.$$

Now the polynomial $a^2(x)$ is represented by the convolution vector

$$\mathbf{a}^2 = \mathrm{conv}(\mathbf{a}, \mathbf{a}) = [0\ 0\ 0\ 0\ 9\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 12\ 0\ 0\ 6\ 0\ 0\ 0\ 0\ 0\ 4\ 0\ 0\ 4\ 0\ 0\ 1]^T,$$

and in general the power $a^r(x)$ is represented recursively by the convolution

$$\mathbf{a}^r = \mathrm{conv}(\mathbf{a}^{r-1}, \mathbf{a}).$$

To analyze totals for r rolls, we need the strictly lower triangular matrix of order $n = 14r + 1$ given by $L_n = (l_{i,j})$ where $l_{i,j} = 1$

"Never give a sucker an even break." — W. C. Fields, 1923 in "Poppy"

if $i > j$ and 0 otherwise. The number of times A beats B with r rolls is given by $\mathbf{a}^{rT}L_n\mathbf{b}^r$. Similarly, the number of times B beats A is given by $\mathbf{b}^{rT}L_n\mathbf{a}^r$. We do not need it, but the number of ties is $\mathbf{a}^{rT}\mathbf{b}^r$. For verification, we can sum these three numbers to get the total 6^{2r}. The table below summarizes the results of summing r rolls for each $r \leq 15$. Notice that ties occur if and only if 3 divides r. This is not surprising once we note that each face of A is congruent to 2 mod 3, on B it is 0 mod 3, and on C we have 1 mod 3. Thus ties can happen only when 3 divides r.

Table 1 shows the probability that the left competitor of each pair wins. For $r \leq 14$, the three possible pairs always form a nontransitive triple. Moreover, it is the original triple $r\mathrm{A} \rightarrow r\mathrm{C} \rightarrow r\mathrm{B} \rightarrow r\mathrm{A}$ for $r \equiv 0$ *or* 1 mod 3 and it is perversely reversed to $r\mathrm{A} \rightarrow r\mathrm{B} \rightarrow r\mathrm{C} \rightarrow r\mathrm{A}$ whenever $r \equiv 2$ mod 3. We certainly might wonder whether this pattern continues for all positive r, but the pattern fails for $r = 15$. Here B dominates both A and C, with C → A. Thus, the first player could select B and, for the first time, transitivity guarantees a winning advantage. Of course the second player can hold his disadvantage to a minimum by selecting C.

Incidentally, the set presented here is about the fifth I investigated. I tried various ways to try to extend the number of nontransitive triples through the largest possible value for r. Earlier sets had their first failures at 3, 9, and 12. Upon finding the present set, I chose to go no further. I do not know if it is possible to find a set that is nontransitive for every possible number of rolls r. Perhaps you can find a better set.

A Nontransitive Variation

Suppose we have a large supply of amber, blue, and crimson dice with the faces given above. A variation of the game would be to let you select any two dice, of like or different colors, then I select my pair. We roll our chosen pairs and compare totals. If a tie occurs we roll over. This means you have six selections available: AA, BB, CC, AB, AC, and BC. We have already seen that B → A and AA → BB. But, when I have second choice, is AA the pair I should choose to gain the greatest advantage over BB? The results can be presented by the directed graph shown in Figure 1. Here we place an arc joining WZ to YZ whenever WZ → YZ. If a pair happens to be perfectly fair, we have no arc joining them. For example, there is no arc between AA and BC because these pairs happen to be totally fair. Moreover, at each vertex we have selected the incoming edge with the largest margin to appear as a green arc. The green arcs identify the best selection for the second player for each possible first choice. Observe that the strongest dominations give rise to a nontransitive cycle among the six possible pairs, namely

$$\mathrm{AA} \rightarrow \mathrm{AC} \rightarrow \mathrm{CC} \rightarrow \mathrm{BC} \rightarrow \mathrm{BB} \rightarrow \mathrm{AB} \rightarrow \mathrm{AA}.$$

The first player, seeking to minimize his opponent's advantage, needs to select either BB or AC, both of which give up only a 53.94% edge.

If we alter the rules to select three dice each, repetitions allowed, Figure 2 shows the strongest dominations for each choice. While each selection wins against certain other selections, and each loses, some selections, AAA for example, never win by enough to be the dominant choice. Here the first player can choose ACC to minimize his opponent's advantage to 52.685% via BBB. Similarly, the strongest dominations are also shown for a selection of 4 dice in Figure 3. Here the first player minimizes his disadvantage by selecting AACC to hold his opponent to 52.127% via AAAC.

We have presented these possible variations of the game to illustrate how difficult it is to predict the best choices for each player.

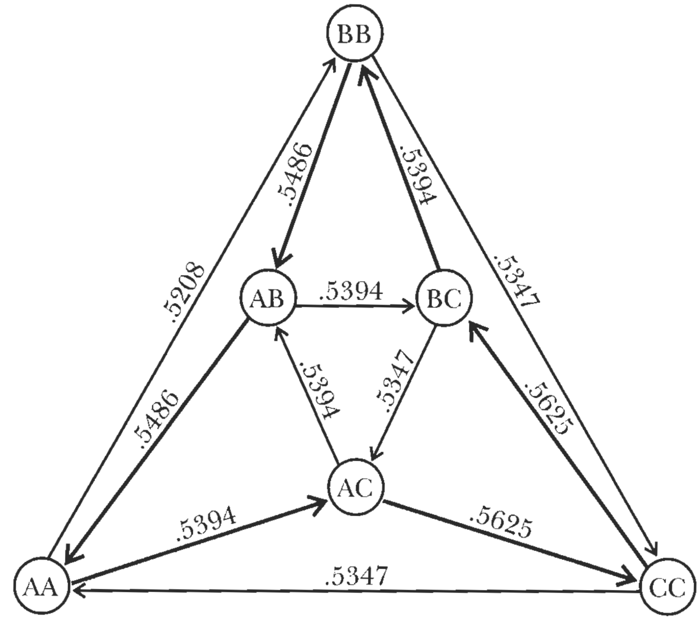


Figure 1. *The domination digraph for all possible pairs of two dice.*

r	A versus B	B versus C	C versus A
1	$\frac{15}{36} \approx 0.41667$	$\frac{15}{36} \approx 0.41667$	$\frac{15}{36} \approx 0.41667$
2	$\frac{675}{1296} \approx 0.52083$	$\frac{693}{1296} \approx 0.53472$	$\frac{693}{1296} \approx 0.53472$
3	0.49328	0.49725	0.49725
4	0.47468	0.47555	0.47555
5	0.50962	0.51533	0.51533
6	0.49658	0.49858	0.49858
7	0.48545	0.48577	0.48577
8	0.50744	0.51004	0.51004
9	0.49773	0.49949	0.49949
10	0.48903	0.49021	0.49021
11	0.50646	0.50808	0.50808
12	0.49821	0.49993	0.49993
13	0.49069	0.49246	0.49246
14	0.50579	0.50711	0.50711
15	0.49846	0.50025	0.50025

Table 1. *Probability that Left dominates Right for the total of* r *rolls.*

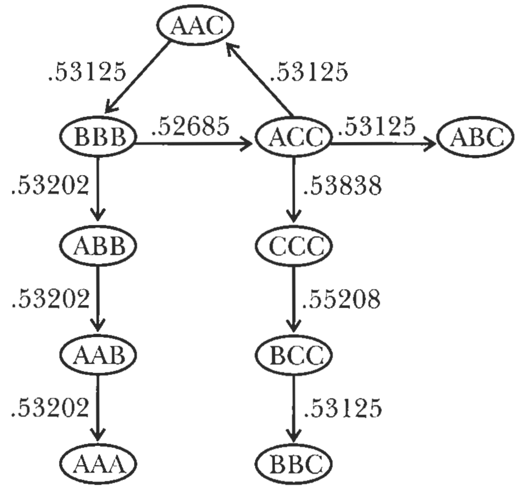


Figure 2. *Strongest domination for sets of three dice.*

Five nontransitive dice

Here is a set of five 5-sided dice, colored amber, blue, crimson, dayglo orange, and evergreen, that have been designed to demonstrate how much optimal choices can vary. Lest you worry about how to construct 5-sided dice, it is very simple. We use cubes, with one side labeled $*$. Whenever the star side lands facing up, the player rerolls.

$$A = \{4, 4, 4, 4, 4\}$$
$$B = \{3, 3, 3, 3, 3\}$$
$$C = \{2, 2, 2, 7, 7\}$$
$$D = \{1, 1, 6, 6, 6\}$$
$$E = \{0, 5, 5, 5, 5\}$$

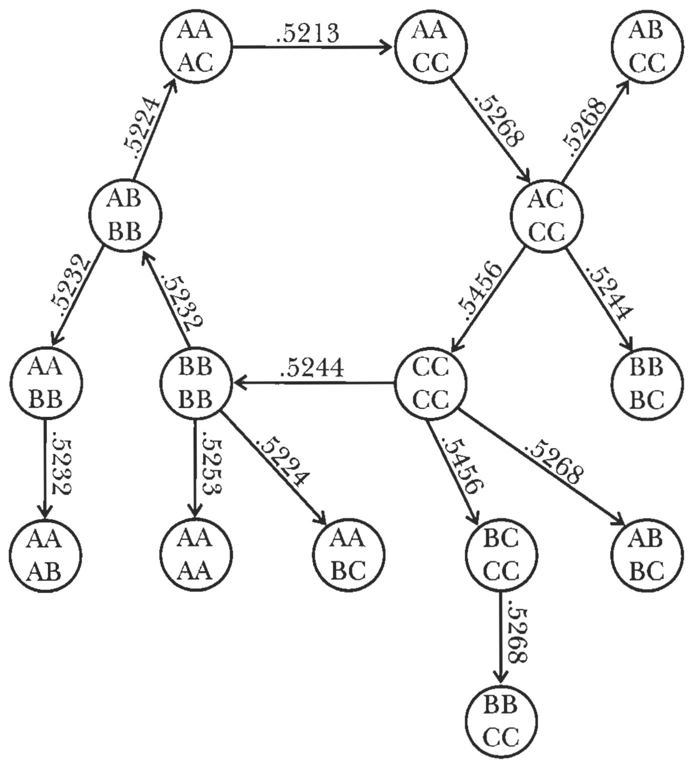


Figure 3. *Strongest domination for sets of four dice.*

By now it should not surprise you that these dice beat one another in a nontransitive fashion. For a single roll the strongest dominations occur in a cycle

$$A \to B \to C \to D \to E \to A.$$

Within this cycle, each domination is by at least 64%. But how does this cycle vary as the number of rolls r increases? When $r = 2$, the dominant cycle has changed to

$$2A \to 2D \to 2B \to 2E \to 2C \to 2A.$$

For $r = 3$, we reverse this to get

$$3A \to 3C \to 3E \to 3B \to 3D \to 3A$$

Next $r = 4$ reverses the original cycle to yield

$$4A \to 4E \to 4D \to 4C \to 4B \to 4A.$$

This is perverse reversal with a vengeance. After the first player selects his die, the second may optimally select any one of the remaining four dice, depending upon how many rolls are intended. And what happens for $r = 5$? Surprisingly we now have a transitive domination sequence

$$5E \to 5D \to 5A \to 5C \to 5B\ .$$

The first person can finally grab the advantage by selecting E. All the second player can do is minimize his losses by selecting D. Curiously, the domination pattern for each value of $r = 6$ to 10 repeats the pattern for $r - 5$.

Many mysteries remain in the realm of nontransitive dice. Why not invent your own and see what surprises you can discover? ■

References

1. M. Gardner, The paradox of nontransitive dice, *Scientific American*, 223, (1970), 110–111.
2. M. Gardner, On paradoxical situations that arise from nontransitive relations, *Scientific American*, 231, (1974), 120–125.
3. R. C. Tenney and C. C. Foster, Nontransitive dominance, *Math. Magazine*, 49, (1976), 115–120.

PHILIBERT SCHOGT

The Wild Numbers

That first night, I set out in high spirits, following the path Riedel had cut into the flanks of the Wild Number Problem. The point that he had reached in 1912, his proof that there were an infinite number of tame numbers, served me as a base camp. From there, equipped with my specialized mountain gear, that is, with Dimitri's new concepts, I could continue my ascent. Every step I took required my fullest concentration; now and then I had to stop to catch my breath, giving me a moment's rest to marvel at the wondrous mathematical landscape all around me.

It seemed like an eternity since I had last ventured into this realm with my undivided attention, free from voices telling me to turn back or urging me to hurry up and get something published. It was like a homecoming, a return to the happiest moments of my childhood, when every new insight made the world bigger and more mysterious, not smaller and more banal, as was so often the case nowadays.

Back then, even the simplest rules and techniques of arithmetic were a source of great joy. In grade two, for instance, Miss Wallace explained to us how to "carry a one" when adding very big numbers, like twenty-seven and thirty-five. When I came home that afternoon, instead of going outside to play (instead of going outside to develop my communication skills, Kate would say), I went upstairs to my room to add. After adding pairs of numbers for a while, I tried adding three at a time and was delighted to discover that at a given point I had to "carry a two." I then decided to go all out, writing down ten rows of numbers consisting of nines only. Just as I had hoped, I soon found myself having to carry a nine, and another nine, and so on. My mother had to call me three times before I finally came downstairs for dinner. Later that evening, I stuffed a sweater into the crack beneath my door so that my parents would not see my light was on, wrote down twenty twenty-digit numbers and proceeded to add them. When I had completed and double-checked the sum, I would have gladly started on an even bigger one had I not been overcome with sleep.

PHILIBERT SCHOGT has a degree in mathematics and philosophy. He lives in Amsterdam, The Netherlands. This is an excerpt from his novel, *The Wild Numbers*.

During the Christmas holidays that year, I spent most of my time up in my room adding large numbers. My parents were once again going through what they called "stormy weather." My younger brother Andrew fled outside; I fled into the serene world of addition, soon too absorbed in my work to pay attention to the screaming, the slamming of doors and the leaden silences that followed. I was high up in the mountains, where my parents' quarrelling sounded as insignificant as the pounding and squeaking of a lumber mill deep down in the valley.

Subtraction, and "borrowing" ones, although presented to us as the opposite of addition and carrying ones, turned out to be much more difficult and subject to peculiar restrictions: Miss Wallace warned us that we were not allowed to take a bigger number away from a smaller one. When I asked her what would happen if we did, she hesitated before answering, a panicky look in her eyes. "Well, let's just say you get zero, all right?" Her answer did not make sense. How could five minus five equal zero and five minus eight equal zero as well? The difference of three couldn't just vanish into thin air. "Stop worrying so much about it, Isaac," she said. "Just let it be zero for now." Her reassurance only made me more anxious. There was no such thing as "for now" in arithmetic. If five minus eight did equal zero, it always had and it always would.

That evening, I gathered all my courage and asked my father the forbidden question: "How much is five minus eight?"

"Negative three," came his voice, god-like from behind the newspaper. He usually did not like being disturbed while reading, but to my delight he folded the paper and, writing in the empty space in an automobile ad on the back page, showed me that there were numbers lower than zero, numbers with a minus written in front of them. I was shocked and thrilled by this new insight. Zero was no longer the absolute bottom of the arithmetical world, but the portal to an arithmetical underworld. It made such a impression on me that when my father laid an arm around my shoulders and told me that he was going to live somewhere else for a while, the news didn't really register.

Five minus eight equalled negative three. Fifteen minus thirty-two equalled negative seventeen. I could not sleep that night, the depths below zero giving me vertigo. Drawn to the edge, at first terrified, I then abandoned myself to falling. I

DON ALBERS

Philibert Schogt and The Wild Numbers

Philibert Schogt

The Wild Numbers is a rarity, a novel about mathematics, which features Isaac Swift, a competent mathematician who dreams of becoming a great mathematician. Solving a famous problem is, of course, a quick path to fame. For a few days, Isaac believes that he has solved Beauregard's Wild Number Problem. There seems to be just one last obstacle standing in the way of success: a student accuses him of plagiarism. During Isaac's short rise and terrible fall, he reveals much about the mathematical enterprise—its competitiveness, isolation, and stimulation. Along the way Isaac encounters a collection of well-drawn mathematical characters, including a venerable and wise senior mathematician, a jealous colleague, and a mathematical crank.

Since 1992, at least four fictional works featuring mathematicians have been published.* For mathematicians, this is a veritable bumper crop. They include three novels and a play. Two of the four authors have advanced degrees in mathematics, and one of them, Petsinis, is a professor of mathematics.

Given the special language and culture of mathematics, the fact that few mathematical novels exist may not be too surprising. The author, Philibert Schogt, brings a background in mathematics to *The Wild Numbers*. Both of his grandparents and an uncle on his father's side were mathematicians. His grandmother was the first woman to study mathematics at the University of Amsterdam. Schogt himself minored in mathematics while completing his MA in philosophy, also at the University of Amsterdam. "Math was always my favorite subject at school," he says, "until in my teens I went through an anti-science phase and my field of interest shifted to the arts, in particular to writing...By writing a novel about a mathematician, I combined my two fields of interest."

Schogt's original plan in 1992 was to write a novel about a mathematician who thinks he has found a solution to Fermat's Last Theorem, a problem which he worked on as a teenager. "I was not sufficiently mathematically sophisticated to experience any real success," Schogt says. "A mathematician friend talked me out of writing a Fermat novel, and that turned out to be a wise move, for while working on *The Wild Numbers*, Andrew Wiles was busy proving Fermat's Last Theorem. The story of Wiles, many would argue, is better than fiction."

Schogt says that, "I wanted the novel to seem like an insider's story, doing justice to pure math and its practitioners. At the same time I wanted to keep it readable for outsiders, but without the reassuring tone of popular science books." Several have commented favorably on the insights Schogt has about the mathematical community, including Sir Roger Penrose, who says, "*The Wild Numbers* provides excellent and entertaining insights into the lives and the ill-understood drives of working mathematicians."

If you're looking for a short novel about mathematicians that includes a bit of romance and even a dollop of violence, then *The Wild Numbers* is for you.

---

* 1992 *Uncle Petros & Goldbach's Conjecture* — Apostolos Doxiadis; 1993 *Arcadia* — Tom Stoppard; 1996 *Leaning Towards Infinity* — Sue Woolfe; 1997 *The French Mathematician* — Tom Petsinis

seemed to be sinking through my bed, my bed was sinking through the floor, the house was sinking into the ground, everything sank into the deep, dark world of negative numbers. And all of a sudden there was my father, in a magnificent purple mantle and with a crown on his head, waiting to welcome me to his new kingdom.

The insights that I was gaining into the Wild Number Problem twenty-eight years later were not as earthshaking as my first acquaintance with negative numbers, though only by a matter of degree. Every step I took, no matter how small, revealed new mountain-tops and unexpected canyons in the magnificent and bizarre region of mathematics first explored by Anatole Millechamps de Beauregard.

This was mathematics at its very best. Unlike in other areas of thought, where knowledge tends to increase gradually, in mathematics the transition from ignorance to understanding is instantaneous and absolute. Either you see it or you don't. But if you do, the new land presents itself in razor-sharp focus, its beauty so intense that you feel you have grown wings and are capable of flying. It is what makes mathematics so addictive. I would not be surprised if there was a biochemical correlate to these flashes of understanding, some kind of opiate that the brain releases into the nervous system every time they occur.

Even Kate, who had a strong aversion to the exact sciences, once experienced the ecstasy of a mathematical revelation. The night that it happened, we fell in love.

Stan called one evening to ask me whether I was interested in saving a damsel in distress. Kate, a good friend of his, was working on her Ph.D. thesis in psychology. Her supervisor had demanded that she take a refresher course in statistics. With the exam coming up soon, she was close to despair. Would I be willing to explain the basic concepts to her? "By the way," he said, "she's kind of cute."

"Cute" was not the first word that came to mind when I showed her into my study the next evening. One would expect some token of gratitude for a stranger offering his help, but instead, her dark eyes flashed accusing looks at me. While unpacking her books, she fulminated against striving for mathematical precision in the domain of human emotions. I was not given the chance to agree with her. Being a mathematician, I was automatically one of the bad guys. The statistical approach to psychology was so revolting to her that she was physically incapable of studying the material. It was masculine thinking at its very worst. Why did we men demand that something be quantifiable before considering it scientific? She had the answer: because we panicked when confronted with matters that defied order, and thus banned them from our world. And why did we panic? Because the most disorderly, formless matter of all was our own pent-up, fucked-up emotionality.

"Why don't you have a seat," I said, sensing that it was pointless to argue with her.

We read through the introductory chapter of her statistics textbook. She was obviously intelligent enough to understand the material, but whenever I introduced a mathematical symbol into my story, she suffered a violent allergic reaction.

"Sigma this, sigma that," she said, waving her arms, "You keep assuming I know what you're talking about!"

"I'm sorry." I explained to her what a sigma was.

"That's not the way you were using it just a minute ago."

"Yes it is."

"Well, you weren't very clear, then."

And so the evening progressed. At three o'clock in the morning, in the middle of yet another one of my attempts to explain something, she threw her pencil down on the table.

"This is pointless. I'm sorry, but I guess I'm just too dumb for this fascinating field of yours."

"Of course you're not!" I cried, fed up with her obstinacy. "And by the way, I'm not particularly fond of statistics either."

For the first time that evening, she smiled.

"Now please give it one more shot."

"All right. For your sake."

"Now look. First you make a column of these numbers, you see?"

She pouted her lip and would not look at me as I went on with my explanation, but at least she no longer objected to every single step.

"And finally, to get the standard deviation, you add these squares, divide by n ..."

"Wait wait, shut up for a second." She studied the figures on the paper with a painful grimace. "So what you're saying is: add up this column, divide by that n over there, and ..."

What took place then was the miracle of mathematical revelation: in a single instant, her dark brooding expression turned into dazzling sunshine.

"I don't believe this! I actually get it!"

Having completed the climb, we threw down our heavy backpacks and wiped the sweat from our brows. We were now standing together on the mountain pass, marvelling at the mathematical landscape. When I saw the panorama reflected in Kate's eyes, I noticed for the first time how beautiful she was. The women I had been attracted to over the years had never understood my passion for numbers, leading me to the conclusion that love and mathematics were mutually exclusive. But now, my adolescent dream of being able to share what mattered most to me with a girl was coming true.

"Unbelievable," she said. "Is this all there is to it?"

I hoped she would not notice the clock on the bookshelf: it was past four.

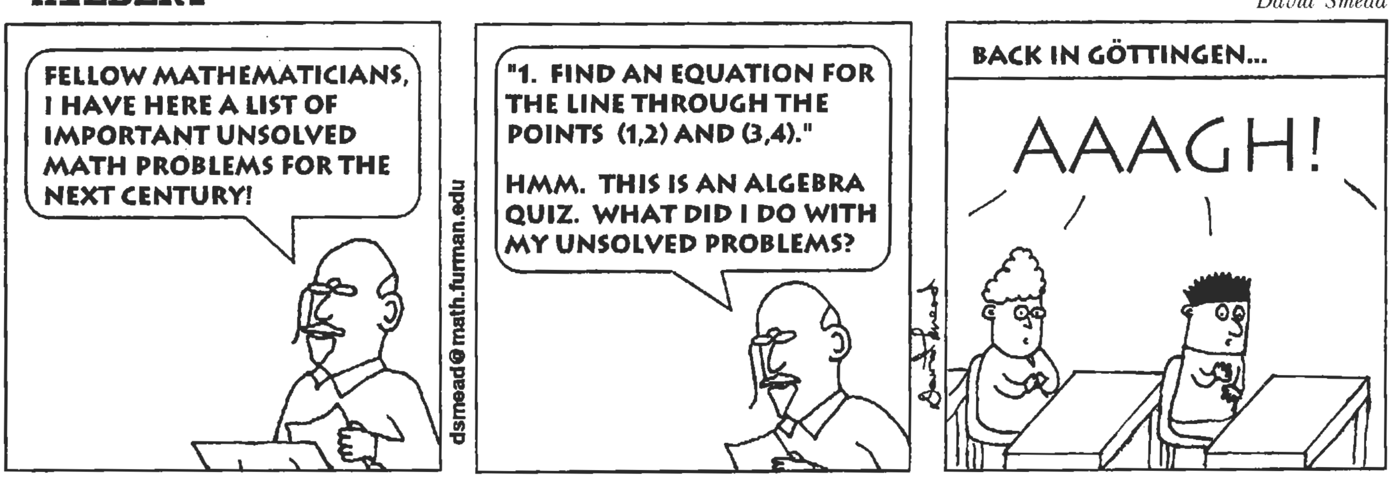
"Give me some more raw data," she said hungrily. "I want to see if I can figure this out all by myself. And don't make it too easy."

When I handed her a new list of numbers, she beamed as if it were a precious gift.

I watched her fingers playing hopscotch on the calculator keys, watched the way she bit her lower lip whenever she scribbled down a result on the sheet of paper, the way she chuckled when the figures still added up after double-checking them. She let her hand rest on my forearm whenever she

HILBERT

David Smead



wanted to know something, and as we looked through her calculations together I could feel the warmth of her cheek near mine. I wondered whether I could get away with a kiss. Later she told me that she had done everything to encourage me. But I didn't dare, and so she kept asking for new problems. We were still sitting at my desk when dawn began to glow through the curtains.

For the first time since that memorable night with Kate, dawn found me at my desk again. I was deeply content in spite of having made little progress. My various attempts to apply Dimitri's method to the problem had been far too reckless, as if I could reach the top by charging straight up the mountain. Every time, I came tumbling back into base camp, dragging an avalanche of mistaken notions down with me. But at least I now knew how not to go about it, and besides, my first inspired night in years was well worth a few conceptual bruises.

To unwind after the long night of hard work, I stepped onto my balcony. It was freezing cold, and my eyes, painfully dry from staring so long at equations on paper, were filled with soothing tears. Down below, a *Chronicle* van rounded the corner with screeching tires and went roaring down the deserted street. It was too early for distinct colors: the city was immersed in uniform blue-grey. Only in the distance, the lights of the television mast flashed red and white, red, white. Kate and I had stood here too, after that night, watching the city slowly come to life, holding hands, kissing.

Several hours later, I was awoken by the telephone. It was my mother, to see if I felt like joining the others on Sunday. This was a recurring ritual: Andrew, Liz and their two children had dinner at her house every Sunday, and once in a while she felt obliged as a mother to invite me too. Even more sporadically, I felt obliged as a son to accept the invitation. The get-togethers were awful: I had to put up with two spoiled brats screaming all evening, while the conversation of the adults — insofar as a conversation was possible — only consisted of long-winded negotiations between grandma and parents concerning which one of them would take which child where at what time. As usual I politely declined the invitation, and as usual my mother didn't insist.

Only when I had hung up did I feel how little I had slept. Five plus four equalled nine. Seventeen minus twenty-eight equalled negative eleven. But instead of clearing my mind, the arithmetical exercise made my head throb. During the few hours of sleep, all thoughts on the Wild Number Problem had congealed into a nagging headache.

In the bathroom mirror, I looked into the dazed eyes of someone who had been alone far too long. "Hello," I said. "Hello, hello." My voice sounded strange, as if it belonged to someone else.

Eating breakfast was a chore: the granola, advertised on the box as being extra crunchy, felt unpleasantly rough on my tongue and made so much noise inside my head that I set the bowl aside and settled for coffee. I was suffering from an all too familiar feeling: a mathematical hangover.

Even when I was a child, my mathematical sprees invariable ended in hangovers. On one such occasion, I had just discovered a mysterious relationship between the square of the sum of any number of variables and the sum of their squares. I was up in my room, diligently writing it down in neat form.

$$(a+b)^2 = 2(a^2+b^2)-(b-a)^2$$
$$(a+b+c)^2 = 3(a^2+b^2+c^2)-((c-b)^2+(c-a)^2+(b-a)^2)$$
$$(a+b+c+d)^2 = 4(a^2+b^2+c^2+d^2)-((d-c)^2+(d-b)^2+(d-a)^2+(c-b)^2+(c-a)^2+(b-a)^2)$$

Years later, I was greatly disappointed to find out that the only mystery lay in my unfamiliarity with certain mathematical laws, the triviality of the relationship being hidden by the inefficient way in which I had expressed it. At the time, however, I was so thrilled by my discovery that my mother was halfway up the stairs before I finally heard her calling my name.

...

As I grew older, I began to wonder whether mathematics really was a passion and not an addiction, a painkiller to dull the ache of unfulfilled desires. The pleasant effects of doing mathematics were gradually weakening, so that ever greater doses were needed, draining all the energy and healthy longing out of me. Was it love or compulsion that made me major in mathematics at university and later choose it as my profession?

To Kate the answer was obvious.

It was in the period that she was living with me that my career began to stagnate. My thesis, a detailed study on Templeton functions with countless openings for further research, had got me the job at the university. But even mathematics is subject to changing fashions, and of late everybody seemed to be losing interest in my area of research. Meanwhile, a former teaching assistant of mine, five years younger than me, had submitted his first article to *Number.* His name: Larry Oberdorfer. When he received the letter of acceptance from the editors, he spent the rest of the day roaming about in the hall. "It's a bird! It's a plane!" he would cry, and then jump into a colleague's office with his arms outstretched. "It's Numberman!"

Although Dimitri insisted that dwindling interest in Templeton functions was a matter of plain bad luck, and Angela warned me not to let myself be intimidated by Larry, I blamed myself for my lack of success; I had not worked hard enough to show the world the value of my work.

In a last-ditch effort, I persuaded Kate to waive the rule "No mathematics after dinner," promising her that my nightly sessions would only be temporary. After weeks of hard work, I was finally close to a publishable result. I had found a theorem which I knew with certainty to be true, but working out the details of the proof required an enormous amount of patience and mathematical technique. There was little reason to celebrate, for I knew in advance that the final result would be too meager for *Number.* And it was unlikely to rekindle anyone's love for Templeton functions. Was it worth the bother? Meanwhile Kate was losing her patience, nagging at me that we never went out anymore, that I was distant, that locking myself up like this was unhealthy.

To appease her, I agreed to go out that next Friday night. The world-famous Deirdre Lindsay Dance Company, which of course I had never heard of, was in town. Before the performance, we went to a fancy restaurant. Throughout dinner, we talked about our relationship, that is, Kate talked about me and I listened. Templeton functions were still buzzing in my head, making it difficult to concentrate on what she was saying, let alone to defend myself.

She had come to the conclusion that I was using mathematics as an escape, as a means to hide from my deeper feelings. I stared at her glassily while she stroked my hand and looked into my eyes with a warm, concerned expression.

"I don't think you have ever stopped to realize how hurt you were as a young child, when your parents got divorced. All you did was go up to your room to add and subtract."

I nodded. Mathematics was a drug, a painkiller. I had thought of that years ago.

"And I don't think you've ever come to terms with your being jealous of Andrew. He got your mother's love, he scored with the girls..."

Yes, That made sense too. If only the noise in my head stopped.

"And now you're upset by Larry publishing an article in *Number*. Don't you see the connection?

I shook my head.

"Larry is your younger brother Andrew all over again."

This time, I felt compelled to react. "Don't be ridiculous!"

Kate smiled and stroked my hand some more. "I hate to play that old psychologist's trick on you, but if it isn't true, then why are you reacting so emotionally?"

"Because you're not being fair."

"Isaac, I am not an adversary. I am only trying to help you. I have a feeling that there is a wealth of emotions that you left behind in your childhood. I know they're inside you somewhere. You can be a very warm and loving person. But you keep shutting yourself off, retreating into the safe, orderly world of abstraction."

"So what do you want me to do? Give up mathematics?"

"Isaac!" she said reproachfully, but I didn't get a real answer.

She continued with her analysis of my personality during dessert and coffee. The more she picked me apart, the warmer her expression became. When it was time to go to the ballet, she was deeply in love with me again.

"I am so happy we finally talked, Isaac," she said, locking her arm in mine as we crossed the street towards the theater. "Aren't you?"

"Uh-huh."

When the lights in the theater were dimmed, she leaned towards me and gave me a passionate kiss. Deirdre Lindsay and her dancers began to prance about on the stage. "No story, no hidden meanings," according to the program, "just an ode to life, natural and pure." But I was left unmoved by what I saw, except for being mildly annoyed by the artificiality of the dancers' smiles. Kate held her hand in mine, our fingers interlaced. Two flaps of human flesh, I thought. Suddenly, the buzzing in my head became louder. If I switched around two steps in my proof, not only would it be greatly simplified, the implications of the theorem would be much farther-reaching! If only I had pen and paper to write it all down. I prayed that I could hold onto my thoughts. I could already feel them slipping away.

In the intermission, I excused myself and hurried to the washroom. Locking myself in one of the cubicles, I groped around in the inside pocket of my jacket and found a pen, yanked a strip of toilet paper from the roll and wrote down as much as I could remember. Switching the steps around was not as easy as I had thought: $f'(x)$ *is an element of*, $f'(x)$ *is an element of*... The gong sounded. Damn! I flushed the toilet and stepped out of the cubicle. Several men by the urinals turned and stared. I wiped the sweat from my forehead with the toilet paper full of notes, then stuffed it into my pocket.

"Isaac, are you all right?" Kate asked me. "You were in there for an awfully long time."

"I'm fine. I'm fine."

...

She fell asleep in my arms. I was wide awake, now only a few minor steps away from cracking the problem. But I would have to get to my study, to pen and paper. Ever so carefully, I lifted her limp arm from my chest and let it fall onto the pillow. She made some smacking sounds with her lips, rolled over onto her other side and went on sleeping. I got out of bed, swept up whatever garments I could find in the dark and tiptoed out of the room.

Soon I was sitting at my desk, wearing pants but no shirt and only one sock. Kate's dress, accidentally brought along too, was now hanging over the chair across from me. At first I cast anxious looks at the door, but then I squared my shoulders and gave the Templeton functions my full attention. Alas, hours of meticulous work ended in deception: there was nothing to be gained from switching around two steps in my proof.

I stared at Kate's dress. She had not even noticed how distant I was during our lovemaking. On the contrary, she thought we had never been so close. Her passion and tenderness had been wasted on me, all because of a bunch of silly mathematical equations. In a delayed reaction, I was flooded with warm feelings toward her.

With tears in my eyes and determined to make more of an effort in the future, I crept back into bed and curled up against her. I wanted to make love to her again, this time with my undivided attention. I laid my hand on her thigh and kissed her shoulder repeatedly, but she was too fast asleep to respond.

It was the last time we came close to being close. The next morning while I was still sleeping, Kate found her dress in my study. Puzzled and upset, she woke me up. When I had confessed my crime she was furious, as if just having found out that in spite of assurances to the contrary, I was still seeing some other woman. ■

DALE BUSKE and SANDRA KEITH

GIMPS Finds Another Prime!

YOU TOO CAN BE A WINNER!! No, this pitch is not coming from Ed McMahon or the Publishers' Clearing House. If you own a 200 MHz (or better) Pentium computer, you too can participate in the Great Internet Prime Search (GIMPS). The Electronic Frontier Foundation is offering \$100,000 to the individual or team which finds the first ten-million-digit prime number, and up to \$250,000 for larger primes. In 1996, George Woltman, using networking software by Scott Kurowski, created the database GIMPS, which coordinates the efforts of more than 8,000 computer users internationally in a practice known as "distributed internet data processing." Woltman's powerful "Prime Net" server distributes and collects work from participating individuals, effectively operating as a single, gigantic parallel processor. Its ambition is to find whether certain large numbers are prime.

In June 1999, GIMPS found its fourth prime, the new world record: $2^{6,972,593} - 1$. While Woltman and Kurowski must certainly share the credit, this prime was plucked by Nayan Hajratwala of Michigan. Like others involved, Hajratwala had downloaded GIMPS software that allowed his computer to do the necessary computations between clicks of the mouse and keystrokes and while running overnight. His prime is 2,098,960 digits long — over 7 miles worth of digits and commas — enough to fill a couple of calculus books! A computer running uninterrupted might have made the discovery in 2–3 weeks, but on idle time, Hajratwala's computer required several months. The previous world-record prime, $2^{3,021,377} - 1$, was discovered in 1998 by a 19-year old student, Roland Clarkson.


Marin Mersenne, courtesy Culver Pictures

Primes seem to be universally fascinating. They can even be hard-working, not merely pretty to look at. Primes helped uncover the "Pentium bug." In 1994, Thomas Nicely, a professor at Lynchburg College, noticed that on a Pentium I, (1/824633702441) 824633702441 is not 1, but 0.999999996274709O2. Nicely had been calculating a series of reciprocals of prime numbers, in part to show that PCs could hold their own with supercomputers. His example used the prime $p = 3 \cdot 2^{38} - 18391$.

Now GIMPS looks specifically for Mersenne primes, primes of the form

$$p = 2^n - 1.$$

It is not too difficult to prove that in order for p to be prime, n must also be prime. So for example, 3 is a Mersenne prime, since $3 = 2^2 - 1$. After that comes $7 = 2^3 - 1$, then 31, and so on. As you can imagine, the numbers on the list get large very quickly. Only 38 Mersenne primes are known, and they have not been discovered in any particular order. GIMPS is a methodical search — the hope being to find some recognizable pattern in the distribution of these primes. It was just such a systematic search that led to the discovery of the Prime Number Theorem. (This theorem says that the number of primes less than a number N approaches $N/\ln N$ as N becomes large.)

People have been fascinated with Mersenne primes since the time of Euclid because of their connection to *perfect numbers*. A number is said to be perfect if it is the sum of its positive divisors. For example, 6 is perfect because $6 = 3 + 2 + 1$. When one writes out the factorizations of the first four perfect numbers as:

$$6 = 2 \times 3,\ 28 = 2^2 \times 7,$$
$$496 = 2^4 \times 31,\ 8128 = 2^6 \times 127,$$

one might just notice, as Euclid did, that these numbers have the special form:

$$2^{n-1}(2^n - 1), \text{ with } n = 2, 3, 5, 7.$$

In Euclid's *Elements* we find a proof of the fact that

A number of the form $2^{n-1}(2^n - 1)$, *where* $2^n - 1$ *is prime, is perfect.*

The proof is straightforward. Since $2^{n-1}(2^n - 1)$ is already in its prime factorization, just write down all possible divisors and add them up. Descartes, in a letter to Mersenne, claimed to possess a proof in the other direction, i.e.,

DALE BUSKE is Assistant Professor of Mathematics and **SANDRA KEITH** is Professor of Mathematics at St. Cloud State University.

that *every* even perfect number is of this form. However, it was one hundred years later that Euler published the first proof of this fact. (See *Euler: The Master of Us All*, by William Dunham for a very clear account of this proof.)

Some earlier mathematicians had wondered: if $2^n - 1$ is prime for $n = 2$, 3, 5, 7, do we get a prime whenever n is prime? When $n = 11$, for example? But in 1536, Hudalricus Regius showed that $2^{11} - 1$ factors non-trivially. Now you may wonder why it took so long to factor $2^{11} - 1 = 2{,}047$, not a very large number. The fact is, the notion of exponent was not even a 17th-century idea. The Greeks had squares and cubes because of their geometry, but they were not operating with any sense of algebra, and the idea of factoring $2^n - 1$ at the beginning of the Renaissance was almost out of the question. Getting a point of view that lets you formulate a question is a big deal, mathematically. And working with numbers in those days was very much second fiddle to geometry.

Although other mathematicians played the leading role in the theory of these primes, they are named for Father Marin Mersenne (1588–1648) whose famous conjecture inspired three centuries of computation, controversy, and theory. Mersenne, a Minim friar educated in France by Jesuits during the time of the Inquisition, is something of an enigma himself, simultaneously described by historians as a kindly mentor (to the youthful Pascal), a gossipy old geezer, a shrewd thinker, a brilliant experimentalist, an advanced planner, an inferior mathematician, and possibly even a case study for psychoanalysis. But surely, in that era, with the scientific thought of many brilliant personae beginning to take root in the skeptical and hostile ground of the Church, a person might be a little of this and a little of that. A religious zealot as a young man, Mersenne eventually came to the conviction that the new science of Galileo was the only true faith — that it alone constituted the only irrefutable evidence of God. Unfortunately for Mersenne, his was a religion with a convert of one. But Mersenne's enthusiasm that "scientific truth will out" caused him sometimes to rub his friends the wrong way. For example, in spite of strict instructions to the contrary, he shared the work of Descartes, his schoolboy chum, with a public which included Descartes' enemies — in those days, Descartes' work might have been perceived as nearly heretical. And six years after Galileo had been condemned to silence by the Inquisition (1639), Mersenne edited and published some of his work. Galileo may have felt qualms, but what a boon to science!

Mersenne's real gift lay in his role as a conduit and catalyst for speeding up the exchange of ideas; he was a latter-day mathematical interface, a networker, a mathematical post office. At a time when mathematics libraries and journals were non-existent, Mersenne's 10,000 pages, or 15 volumes of correspondence, brought together the thinking of the greatest mathematicians of the day. (Consider for a moment the scale on which GIMPS continues that networking legacy!) Mersenne also created a discussion club in Paris and helped organize the first attempt, albeit short-lived, of the Paris Academy. Somewhat categorically however, he declared that experimentation was the only profitable approach to understanding the laws of nature. And being stronger on guesswork than supporting argument, he declared (although he could not prove) that the circle could not be squared, that sound is a mode of motion, and that the pendulum could be used as a timing device. His observations and experiments filled volumes, but only his guess about perfect numbers has survived obsolescence. That guess was, in fact, incorrect.

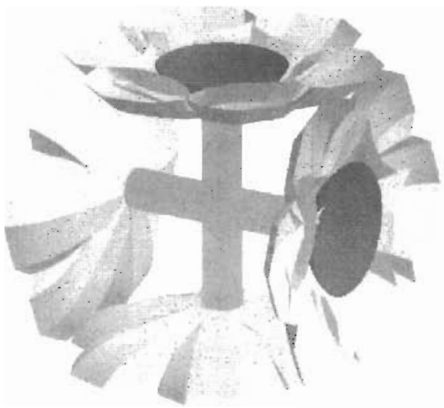
In 1644, Mersenne made the grand conjecture that

> $2^n - 1$ is prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127$, and 257, and composite for all other positive integers $n < 257$.

Now, $2^{257} - 1$ is a tremendous number, and one might just wonder how Mersenne the experimentalist came up with his idea. Mersenne knew that Fermat, a mathematician of no small brain power, had made an interesting discovery:

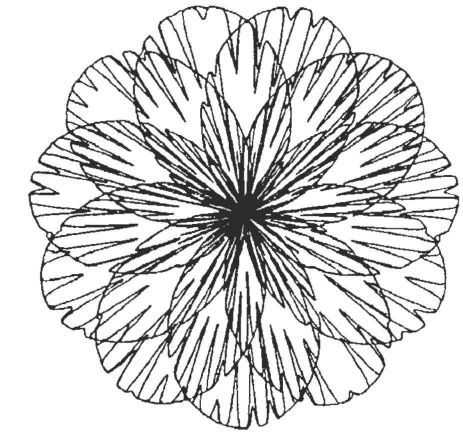
> If q is a prime, and p is a prime divisor of $2^q - 1$, then $p - 1$ is a multiple of q.

So in 1640, when Bernard Frenicle asked Fermat, through Mersenne, a question equivalent to asking whether $2^{37} - 1$ was prime (perhaps as a test), Fermat could show that it was not. How did this work? Since $q = 37$ is prime, any prime divisor p would have the form $p = 37k + 1$, so one can merely check out the primes of



Congratulations to

Nicole Januszewski (St. Cloud State University) and Temple Fay (The University of Southern Missippi) for their winning entries in *Math Horizons*'s Final Exam Contest of parametrizing the perfect flower. Nicole's three-dimensional buttercup and Temple's two-dimensional chrysanthemum won them *Math Horizons* t-shirts.



---

this form. Indeed, one soon finds that 223 (= $6 \times 37 + 1$) is a factor of $2^{37} - 1$. Fermat's method allowed Mersenne to dispose of many other smaller prime candidates without too much effort.

Now consider the following list:

$$5 = 2^2 + 1$$
$$7 = 2^2 + 3$$
$$17 = 2^4 + 1$$
$$19 = 2^4 + 3$$

Do you see a pattern? Then perhaps 67 = $2^6 + 3$ could be a prime, and possibly, even 257 = $2^8 + 1$? Whatever his thinking, and we will probably never know what it was, Mersenne must have had a "powers of two" thing going. Consider the primes 31 and 127 in his conjecture: these can be represented as $2^5 - 1$ and $2^7 - 1$ as well.

So it was big news in 1876, over 200 years after Mersenne's conjecture, when Edouard Lucas showed that the number $2^{67} - 1$, on Mersenne's list, was actually composite. He used a strategic algorithm devised by Lucas (1870s), later to be improved by Lehmer (1930); GIMPS now uses a variant of this algorithm.

> For p odd, the number $2^p - 1$ is prime if and only if $2^p - 1$ divides $S(p-1)$ where $S(n+1) = S(n)^2 - 2$ and $S(1) = 4$.

It didn't take much longer to show that $2^{61} - 1$ (which Mersenne had overlooked) was prime. But because uncorking the factors of a large number is far more difficult than showing it prime, there didn't seem to be much hope, even in the early 1900's, for factoring the awkward $2^{67} - 1$. Enter the algebraist Frank Cole, whose talk, in an October 1903 meeting in New York of the American Mathematical Society, is described by E. T. Bell [1]:

> Cole had a paper on the program with the modest title "On the factorization of large numbers." When the chairman called on him for his paper, Cole — who was always a man of very few words — walked to the board and, saying nothing, proceeded to chalk up the arithmetic for raising 2 to the sixty-seventh power. Then he carefully subtracted 1. Without a word he moved over to a clear space on the board and multiplied out, by longhand, 193,707,721 × 761,838,257,287. The two calculations agreed. Mersenne's conjecture — if such it was — vanished into the limbo of mathematical mythology. For the first and only time on record, an audience of the American Mathematical Society vigorously applauded the author of a paper delivered before it. Cole took his seat without having uttered a word. Nobody asked him a question.

Cole's method had been to sift through quadratic residues [2]. The story goes that when asked, "how long did it take you?", he replied, "two years of Sundays!" As of 1947, the conjecture of Mersenne was corrected to assert:

> $2^n - 1$ is prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107$, and 127 (Note: 61, 89, 107 added and 67 and 257 omitted.)

Are there infinitely many Mersenne prime numbers? Is there a pattern about them? Is there such a thing as an odd perfect number? And what is it about these questions that continues to fascinate us? Since our own computers have been doodling away drawing rotating squeegees anyway, we decided to take part in the search ourselves. After all, in the time it took us to think how to spell "squeegee", our computer has performed tens of thousands of calculations. In fact, we can announce with pride that we are the individuals who recently discovered that $2^{7,121,141} - 1$ and $2^{7,192,501} - 1$ are not prime. No doubt about it, prime hunting can become an obsession: future generations, beware! ■

The authors owe many thanks for the insights provided by Fernando Gouvêa and Robert Burn.

References

1. Eric Temple Bell, *The Queen of Mathematics*. In: James R. Newman (ed.), Volume One of *The World of Mathematics*. Simon and Schuster, New York, 1956, p. 503.
2. Eric Temple Bell, *The Last Problem*, The Mathematical Association of America, 1990.

For more information on starting a distributed computing project like GIMPS, visit the following web sites:

- `www.mersenne.org/prime.htm`
- `www.setiathome.ssl.berkeley.edu/`
- `www.distributed.net/`
- `www.cecm.sfu.ca/projects/pihex/pihex.html`

Three Estimation Challenges
Penny Loss, Class Size, Drug Use

Pinching Pennies

Do you pick up pennies? Most people, it seems, don't bother anymore—and with good reason. It's hard to see how picking up a penny or two will have any material effect on your life, unless you're just short of the cash you need for that doughnut you'd like to buy.

One way to decide if it's worth your time to pick up pennies is to perform the following thought experiment: imagine that there are an unlimited number of pennies scattered all over the ground, far enough apart that you could only pick up one at a time. Estimate how long it would take you to bend down for each coin, then figure how much money you'd collect in an hour. Would you be willing to do such menial labor for the hourly wage you've computed?

In recent years there have been several calls to abolish the penny entirely. Reporter John Tierney, writing in the New York Times [1], notes that the United States Mint is producing more than ten billion pennies a year, most of which are destined for oblivion. They make a one-way trip to penny jars, sock drawers, piggy banks and the spaces between couch cushions. Tierney points out that the U.S. Mint is working around the clock to re-supply banks that have run out of pennies, as according to the Mint two-thirds of the cents produced in the last 30 years have dropped out of circulation.

How the Mint estimates the rate at which pennies disappear is not entirely clear, but it is based on comparing the number of one cent coins they supply to the Federal Reserve Banks to the number that these banks take in from commercial banks.

You can estimate the attrition rate yourself simply by keeping track of the dates of pennies you come in contact with. Table 1 shows how, using one person's actual sample [2]. Notice that the great majority of coins that were encountered are of very recent vintage.

First we need to create a mathematical model of the situation. Such modeling often proceeds by initially considering a very simple, stripped down model, and then successively building more features of the problem into the model. So let's start with the following very basic model: assume that there is a fixed probability p that a penny will disappear in any one year. We wish to estimate p. Also assume that the same number N of pennies is made every year.

Let $q = 1-p$ be the probability of a penny remaining in circulation from one year to the next. Then the expected number of pennies with the date $1999-k$ still in circulation in 1999 is Nq^k, for $k = 0, 1, 2, \ldots$, so the numbers of coins per year in a collection of pennies should decrease approximately geometrically. The chance that a randomly encountered penny is from the year $1999-k$ is then

$$P(k) = Nq^k \div \sum_{i=0}^{\infty} Nq^i = q^k \div \left(1/(1-q)\right) = pq^k$$

(technically, this requires that pennies have been produced infinitely far back in time). A set of probabilities of this form is known as a *geometric distribution.*

Readers who have had a course in mathematical statistics may recall that the optimal estimate of p for a geometric distribution, obtained by the method of

1999	17	1988	5	1977	2	1966	0	1955	0
1998	16	1987	7	1976	3	1965	1	1954	0
1997	16	1986	6	1975	3	1964	1	1953	0
1996	8	1985	8	1974	2	1963	0	1952	1
1995	8	1984	2	1973	0	1962	1	1951	0
1994	7	1983	5	1972	1	1961	2	1950	0
1993	6	1982	9	1971	1	1960	2	1949	0
1992	9	1981	6	1970	0	1959	2	1948	0
1991	2	1980	3	1969	1	1958	0	1947	0
1990	4	1979	5	1968	3	1957	0	1946	1
1989	7	1978	4	1967	1	1956	1	1945	1

Table 1

MARK SCHILLING is Professor of Mathematics at California State University, Northridge.

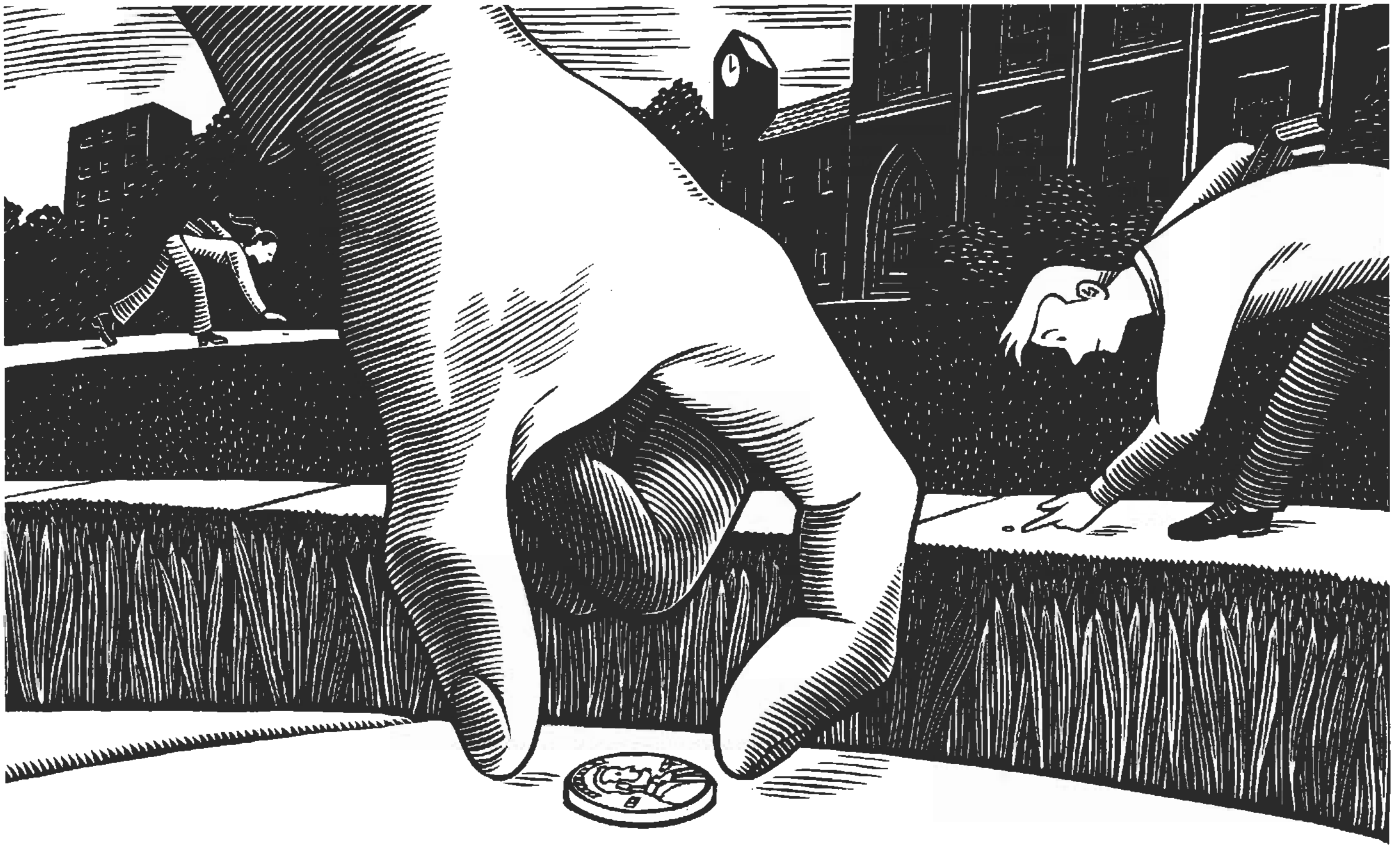


Illustration by Greg Nemec.

maximum likelihood, is $\hat{p} = 1/\bar{k}$ where $\bar{k}$ is the average age (in years) of the coins in the sample. For the data above we find $\bar{k} = 12.02$, so we estimate the attrition rate p as 1/12.02 = .083 = 8.3%. This answer makes intuitive sense—if one out of every twelve pennies is removed from circulation each year, then it seems reasonable that the average time in circulation is twelve years.

A source of inaccuracy in the model above is that the number of pennies made each year has fluctuated significantly over time, growing from about half a billion coins per year after World War II to as high as 16.7 billion in 1983 before dropping to 10.2 billion in 1998. Also, the rate of production in the years 1945–6 was far higher than in the decade following, which helps to explain the presence of these dates in the sample above. We can refine our model to take these variations in production into account.

Let N_k be the number of pennies produced in the year $1999 - k$. The modified probability distribution of penny dates now becomes

$$P(k) = N_k q^k \div \sum_{i=0}^{\infty} N_i q^i$$

for $k = 0, 1, 2, \ldots$. Finding the maximum likelihood estimate of p under this model turns out to be a complex numerical problem. A simpler approach is to normalize each of the observed counts to what they would have been if the production for each year had been the same—say, ten billion per year. (Of course these adjusted counts will no longer be integers.)

Recomputing $\hat{p} = 1/\bar{k}$ using these adjusted counts yields a much smaller estimated rate of penny loss of 5.0% per year, based on an average penny life span of 20.2 years. This figure is quite close to the United States Mint's own estimate of 5.5%. The lower estimated attrition rate reflects the fact that the decrease in penny counts by year is not just due to pennies dropping out of circulation, but also because far fewer pennies were produced many years ago than recently, a fact not incorporated into the first model.

We conclude that the second model is a much better description of reality than the first. A still more sophisticated model would allow the rate of penny loss to vary with time. Substantially more data would be required in order to employ such a model, however, due to the need to estimate each year's attrition rate rather than just one.

A Size Paradox

What is the average class size at your college or university? That's a straightforward question, right? The registrar could easily determine this figure from a list of enrollments for each class in a given term (as of a specified date). In fact such figures are often computed and used for publicity and recruitment.

Imagine, for example, that you attend a small college that has twenty classes, of which ten have enrollments of thirty each, eight are small lab classes with ten students in each, and two are large lecture classes each containing 100 persons. The registrar computes the average class size as the total class enrollment divided by the number of classes, or (10 × 30 + 8 × 10 + 2 × 100) ÷ 20 = 580 ÷ 20 = 29.0 students per class.

As with the penny loss problem, you could estimate this value yourself from the information you encounter personally. The obvious approach is to simply average the sizes of the classes you are enrolled in. You could improve your estimate by asking a couple of friends to give you their class counts as well (choosing people with different majors from your own).

Suppose to get perfect accuracy you decide to distribute a survey to every class at the college, asking each person in the class to give the class enrollment. When the results are compiled, you find that the average class size is 35.5. When your survey result is reported in the school newspaper, there is grumbling in the registrar's office that lots of students must have given inaccurate answers. But did they?

Notice that the number of surveys returned in each class (assuming a 100% response rate) is equal to the class size. Thus there is an *overrepresentation* of the large classes and an *underrepresentation* of the small classes in the survey. Specifically, the average class size computed from all 580 responses is $(10 \times 30 \times 30 + 8 \times 10 \times 10 + 2 \times 100 \times 100) \div 580$, which is indeed 35.5.

No matter what the distribution of class sizes is at a school, the students' average must always be at least as large as the registrar's average. You might wish to try and show this. (Hint: use the Cauchy-Schwarz inequality.) Obviously universities prefer to report the registrar's average. Which average is the most relevant to prospective students?

Surveying Sensitive Questions (Reprise)

In the last column of *Chance Encounters* [3] a discussion was given of how survey researchers can gather honest responses to sensitive questions. The explanation I provided there was in error. The statement given was "For example: 'Have you taken illegal drugs during the past twelve months? Toss a coin and answer truthfully if the coin comes up heads, answer 'No' if the coin turns up tails.'"

Clearly this protocol would reveal some drug users, namely all those who answer "Yes." What I had intended to write was "Toss a coin and answer truthfully if the coin comes up heads; if the coin turns up tails, toss the coin again and answer Yes if the coin gives heads and No if the coin gives tails."

This method preserves confidentiality in that "Yes" answers can be due simply to the result of coin tosses rather than to actual drug use. But there is a cost for the researcher: about half of the responses represent statistical "noise," with no way to distinguish these responses from the ones actually containing answers to the drug use question. How much accuracy does this lose?

For an ordinary survey, the proportion p of the population with a certain trait (e.g., illegal drug use) is estimated by the proportion $\hat{p}$ of the survey respondents with that trait. The standard deviation of this estimate is easily shown to be $\sigma = \sqrt{p(1-p)/n}$ For example, if $p = 50\%$ of the population owns a computer, then for a survey of 400 randomly selected individuals $\sigma = 2.5\%$. By the Central Limit Theorem, there is about a 95% chance that the proportion $\hat{p}$ of the sample who report that they own a computer will differ from p by less than $2\sigma = 5\%$.

For the confidentiality-preserving survey, in which half of the n surveys have relevant answers while the other half have answers based only on a coin toss, p is estimated not by $\hat{p}$ but by $2\hat{p} - .5$, and the standard deviation of this estimate turns out to be

$$\sigma^* = \sqrt{\frac{2p(1-p)+.5}{n}}.$$

(Readers who have had a course in mathematical statistics can try to verify these two facts.)

Clearly σ^* is much larger than σ for a given n. Comparing σ to σ^* we find that n must be at least four times larger in order for a confidentiality-preserving survey to achieve the same sampling error as an ordinary survey (see if can you show this). The price of preserving confidentiality, while likely to at least yield truthful answers, is large indeed. ■

Endnote

The efficiency of the confidentiality-preserving survey method can be increased by reducing the percentage of respondents who will not be asked to answer the sensitive question. For example, the instructions could be changed to "Toss a coin *twice* and answer truthfully if the coin comes up heads *at least once*. If the coin turns up tails *both times*, toss the coin again and answer 'Yes' if the coin gives heads and 'No' if the coin gives tails." Unfortunately the confidential nature of the survey is eroded by such a scheme, in that the chance that someone actually uses illegal drugs (for example) given that they answered "yes" can be quite high. Exactly how high can be determined from Bayes formula, which can be found in virtually any introductory probability textbook.

References

1. John Tierney, "Opportunity presents itself: kill the penny," *New York Times*, August 12, 1999.
2. Chance News 8.07 (`http://www.dartmouth.edu/~chance`).
3. Mark Schilling, "Aliens, Asteroids, and Astronomical Odds," *Math Horizons* (Chance Encounters), November 1999.

The Duplicity of Two

As we round out the second millennium, and begin a second year of the WordWise column (two reasons!), we thought it would be fitting to take a glance or two at the word "two" itself.

One, two, buckle my shoe ... two is the first number that a child learns. And well it should be: the child can point to pairs of eyes, ears, feet, and shoes. In contrast, three, the next largest whole number, is not so biological, while zero and one are actually rather subtle. As the German scholar Karl Menninger has put it, "The number two is a frontier in counting" (Menninger, 1969, p. 15).

We were not surprised, then, to find roots for "two" playing many roles in our language—although some of our findings were quite unexpected.

One Word Twigs Off From Another

Our word "two" stems from the Germanic *twa* or *twai,* also the basis for our word "twig." The etymology of "twig" suggests a branch forking into two pieces. The same root has also given us a host of related words:

twice — two times
twenty — two tens
twain — pair or couple
twins — two children born at once
twilight — two lights, daylight and evening, that we see at dusk
betwixt and between — in the middle of two things
twist — to wrap two filaments around one another
twine — string made of two strands twisted together
twill, tweed — fabric patterns made with doubled thread

Even our English counting word "twelve" has "two" hiding in it: it derives from the German *twa lif* meaning "two left." When you group by tens, twelve has two left over. The same reasoning applies to "eleven," from *ain lif,* "one left."

RHETA RUBENSTEIN and RANDY SCHWARTZ are professors of mathematics at Schoolcraft College in Livonia, Michigan.

The Germanic *twa* twigged off from an even older Indo-European stem, *dwo* ("two"). In Greek this became *di, duo*. Consider our word "diploma," the certificate sought by college students everywhere as proof of all their hard work. The Romans adopted the Greek word *diploma,* literally meaning "doubled" or "folded," to refer to a bound pair of bronze tablets, on the inner surface of which a veteran soldier's privileges would be enscribed. In time, "diploma" was applied to many other types of certificates, and it gave us our word "diplomat," someone certified to speak for a state.

Among the many other English "two" words related to the Greek *di, duo*:

diptych — painting on two panels hinged at the center
dichromatic — having two colors
dichotomy — division into two categories
dilemma — two-choice decision
dihedral — edge or angle formed by two planes
duo, duet — pair of performers
dual — paired, twinned
duel — fight between two antagonists, from *duellem* which also gave the Romans their term *bellum,* "war"
dyad — a pair
deuce — "two" in cards, dice, tennis and other games
duplicate — second copy
duplicity — the coexistence of two sets of motives
duplex — twofold, such as two dwellings in one frame, or two directions of transmission over one cable
double — folded in two
doppelgänger — ghostly double of a living person
doubt — the condition of being of "two minds"
dozen — a group of twelve
duodecimal — base-12 numeration system (*duo* + *deca* = 2 + 10)
dodecagon — polygon with 12 sides
dodecahedron — polyhedron with 12 sides

We Challenge You to a "Dual"

There is an additional mathematical connection between the terms "dual" and "dodecahedron." A regular dodecahedron is a solid with 12 identical plane faces, each face being an equilateral and equiangular pentagon. The centers of these pentagons form the vertices of a regular icosahedron, a solid

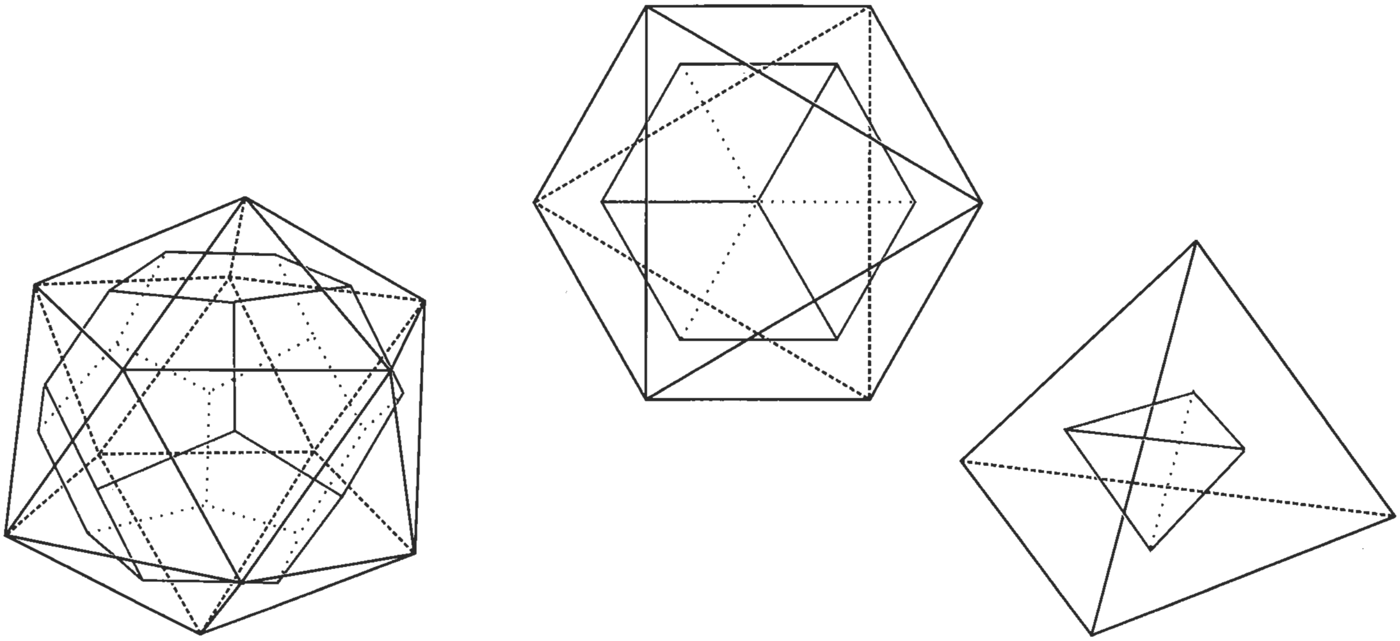


Figure 1. *Among the five Platonic solids, the dodecahedron and icosahedron are dual (left), as are the cube and octahedron (center), while the tetrahedron is self-dual (right).*

with 20 identical plane faces (from the Greek *eikosi*, "twenty"), each face being an equilateral triangle. Conversely, the centers of the faces of the icosahedron form the vertices of a new dodecahedron. The icosahedron and dodecahedron are said to be dual to one another, since they are "paired" or "twinned" in this manner. All of the Platonic solids are dual in this sense (see Fig. 1).

The word "dual" is often used to refer to such mathematical twins. In the study of projective geometry, points and lines are conceived as "dual" to one another; for example, just as two distinct points determine a line, so also two intersecting lines determine a point. In the field of operations research, every linear programming problem whose objective is to maximize a given function subject to a given set of constraints has a natural twin, a "dual" problem whose objective is to *minimize* a related function. Although the final answer to the two problems is identical, the process of finding this answer is often easier for one of the dual problems than for the other, furnishing a very useful shortcut. The Duality Theorem of Gale, Kuhn and Tucker (1951) established this central fact of linear programming.

The root *duo-* can also be found in the name of one of the oldest algorithms in mathematics. Duplation and mediation, a phrase derived from roots meaning "double" and "middle, halfway," is an ancient method of multiplying a pair of numbers. The algorithm, apparently used in ancient Egypt and also known today as the Russian Peasant method, is based on repeatedly doubling one number and halving the other until a one results. When any whole number is divided by two, not coincidentally there are exactly two types of results, depending on whether the dividend is even or odd. See, then, whether you can reconstruct how duplation and mediation works based on the following illustration (discussed by Sgroi 1998: 81–85), in which 26 is multiplied by 55.

~~26~~	~~55~~
13	110
~~6~~	~~220~~
3	440
1	880
	1430, answer

Getting Ambitious About "Bi-"

Just as the Greek root for "two" is *di-* or *duo-*, the Latin is *bi-* or *bis-*, the basis for such words as:

bicycle — two-wheeled conveyance
biped — two-legged animal
bigamy — having two wives
bicameral — having two chambers, like the U.S. Congress or the human brain
bicentennial — two-hundredth anniversary
combine — put two things together
bias — fabric cut across the grain, or a statistical sample that is slanted, from a root meaning "looking two ways"
balance — a scale with two arms
biscuit — an item that has been twice cooked (think of the French for cook, *cuit*) In German we have the same roots to form *zwieback*, a cookie that is twice baked!
bisque — a thick broth that has been thoroughly or twice cooked
bis — a European call at the theater for an encore, or "second performance"; also, on European roadsigns, an indicator for an alternate route

In mathematics, expressions having two terms, such as $x + y$ or $4x^2 - 3y^3$, are called "binomials," from the Latin *bi*, "two" + *nomin* "name, term" as in "denominator." Can you explain why powers of two keep appearing when we add up the "binomial coefficients"?

$(x+y)^2 = 1x^2 + 2xy + 1y^2; \qquad 1 + 2 + 1 = 4 = 2^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3; \quad 1 + 3 + 3 + 1 = 8 = 2^3$

$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4;$

$1 + 4 + 6 + 4 + 1 = 16 = 2^4$

Suppose that a sports team goes through a streak of wins and losses: WWWWWLLLLLWWLLLWWWW. Since there are two letters involved, this is called a "binomial sequence." The odds that such a streak will occur can be calculated using the "binomial distribution" formula.

Further mathematical occurrences of *bi-* include:

bisect — to cut into two parts
binary number — a base-two number; each digit is called a binary digit, or "bit" for short
binary search — an algorithm that searches a list by repeatedly chopping it in two
biconditional — an *if...then...* statement that is true in both directions: "If A, then B" and "If B, then A."
bimodal — said of a statistical distribution having two modes, or predominant values
bijection — a function that is both injective (one-to-one) and surjective (onto)
bifurcation — a function or graph that splits into two branches like the prongs of a fork

Can you guess what a "birectangular" triangle would be? In spherical geometry it is a triangle with two right angles! (See Fig. 2.)

We also find *bi-* in "billion," although the billions of earthlings vary in how they construe this word. Originally "billion" was intended to refer to a number with twice as many zeroes as in one million. Since a million is 10^6, a billion should be 10^{12}, and so it is in the British Isles. The French and Americans, however, were concerned that 10^9 was left without a name, so they moved "billion" back to 10^9. So beware! If you are traveling in England, you need a million (not a thousand) millions to make one of their "billions"!

The Greek *amphi* and the Latin *ambi* mean "on both sides" or "on all sides." More "two words" derive from these:

amphibian — animal that lives both on land and in water
amphora — ancient Greek vase with two handles

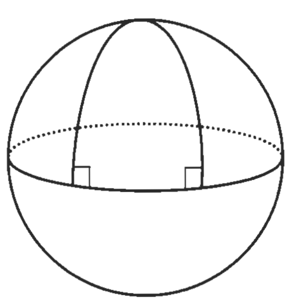


Figure 2. *On a sphere, a birectangular triangle has two right angles.*

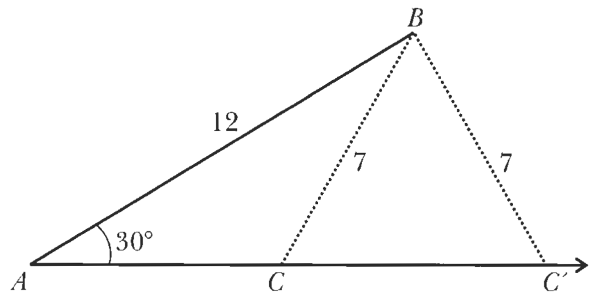


Figure 3. *Side-side-angle (SSA) is an ambiguous case in geometry.*

amphitheater — double theater, two semi-circular banks of seats.
ambidextrous — using both hands with ease
ambition — an ardent or zealous moving about, from *ambire*, "to go back and forth"
ambivalent — feeling both attracted to and repulsed by something
ambiguous — having two (or more) meanings

In trigonometry, if you are given the lengths of two sides of a triangle and the measure of one of the two angles not between them, there might be two different triangles satisfying these conditions. For this reason, the case SSA (side-side-angle) is called "ambiguous" (see Fig. 3).

Redoubling Our Findings

Because the concept is so important in our world, "twoness" is expressed in many different forms. Here are some other words whose roots are worth exploring:

second — derives from the same root as "sequel," suggesting "following or falling behind in order."
pair — from a root meaning "equal," the same root that gave us "peer," a person of rank equal to another, and "par," an accepted standard value, as in golf.
couple — from a root meaning "to fasten together." From "couple" we also get "couplet," two successive lines of verse forming a unit.
yoke — from a root meaning "to join," this wooden bar joins two draft animals so they can be driven together.
brace — a support that fastens two pieces together; or a pair of gamefowl ("a brace of partridges").
half — from *healf*, related to Latin *scalpere*, "to cut." When children talk about "the bigger half," their misunderstanding parallels this origin, which focused more on the number of pieces than on their sizes.
semicircle — from the Latin *semi-*, "half"
hemisphere — from the Greek *hemi-*, "half"

And mathematics is certainly not the only science replete with such words. You might enjoy hunting down the twoness in the meanings of these bits of jargon:

Physics: dipole; binocular; bifocal; bifilar pendulum
Chemistry: deuterium; DDT; diatom; dimer; amphoteric

Biology: bicuspid; bivalve; binomial nomenclature; dicotyledon; diploid; duodenum
Psychology: bipolar disorder; ambiverted
Linguistics: bilingual; diglossia; diphthong; gemination

Why is it that "two" has so many faces- *bi-, di-, twi-, ambi-*, and others? It must be that the concept of "twoness" is so basic that its uses diverged, like twigs, at many points in the growth of the tree of language. With your new knowledge of roots meaning "two," you shouldn't be surprised to learn of the term used by etymologists for pairs like "two" and "duo," which derive ultimately from the same source but have changed in form: they are called "doublets"! ■

References

1. Karl Menninger, *Number Words and Number Symbols: A Cultural History of Numbers* (translated by Paul Broneer), Dover Publications, 1969.
2. Maxwell Nurnberg and Morris Rosenblum, *All About Words: An Adult Approach to Vocabulary Building*, New American Library, 1966.
3. Steven Schwartzman, *The Words of Mathematics: An Etymological Dictionary of Mathematical Terms Used in English*, Mathematical Association of America, 1994.
4. Laura Sgroi, An Exploration of the Russian Peasant Method of Multiplication, in *The Teaching and Learning of Algorithms in School Mathematics*, 1998 Yearbook of the National Council of Teachers of Mathematics.

Continued from p. 9.

and because I have learned at least some of their language. I think Alex Grossmann played a very important role that way. I have met Jean Morlet several times, I think he's a very interesting man, but I find it very hard to talk with him. I mean, of course, not talking socially, but to really understand his ideas. Because it's not even that I can see that here's an idea and I know that I don't understand the formal mathematics, it's that I don't even understand the idea: that I don't even notice or can't tell if there is something there or not.

"The problem is you don't know what are ideas and what aren't: he probably knows the different layers of what he's saying, but for me it's impossible. Alex Grossmann can talk to him. It's good to have a chain of people. I think so. And I think that's the role that I played. In some sense you could say that I didn't discover anything that anybody didn't know because there's this one aspect of the mathematical roots, but then there's the other one which has to do with an algorithm for implementing the whole thing. If there wasn't an algorithm, then none of this would be happening anyway. But that algorithm existed in electrical engineering: it's called sub-band filtering. There was no connection with any of the pure mathematics, and I don't know that that connection ever would have been made if it weren't for this chain of people. I mean, once this connection was made, then you had mathematicians interested in hearing about the algorithms and electrical engineers interested in hearing about the mathematics.

"I'm a mathematician. I feel like one. I feel like a mathematician, but I am very much motivated by applications. I like to go off on a mathematical tangent, but I like to get back to applications. So in that sense, I'm an applied mathematician. At one time, and still today for some people, applied mathematics meant only certain types of results obtained by solving certain types of partial differential equations. I'm not that type of mathematician at all. So I very much feel I'm an applied mathematician, but what I apply is functional analysis, rather than PDE theory. Actually, there's no such field as applied mathematics: I think there are subfields within mathematics and that, as a mathematician, you always really like it when different subfields get into contact with each other. I think that virtually all of these mathematical subfields can have contacts with applications, so in some sense I'm an applied harmonic analyst, not an applied PDE person, but one can just as easily be, say, an applied number theorist.

"Ideally, I think there's an important place for pure mathematics and an important place for pure mathematicians; I see my role as identifying and bringing to more pure mathematicians than myself very interesting problems coming from applications. I think that's an important role to play and that it is good for pure mathematics. There was a while when pure mathematics wasn't open to this, but I think that mathematicians are starting to open up more. It's very important to remember that whole fields in pure mathematics have come from applications. That doesn't mean that all the pure math that was done in that area can therefore be described as applied. No, it's just mathematics. But it also suggests that it's very well possible for other fields of mathematics to start to be fostered by applications. I mean applied mathematics is not just learning some nice mathematics and not being upset by getting your hands dirty on some problems where things are not as neat but they will have an application—it's also identifying opportunities for mathematical thinking, which can lead to other fields of mathematics. I don't know which other fields. I can't predict. I mean it's the ones that you cannot predict that are the most interesting."

A Life in Mathematics

Problems are the lifeblood of mathematics and Daubechies, like most mathematicians, has several going at once. All the ones she tells us about come from real world applications. It's quite clear listening to her explain her problems that she is a phenomenal teacher — the explanations are so clear and the problems sound so exciting that we're itching to get out of the interview and get to work on them.

"Before this meeting I was at a molecular biology meeting. I got really interested in people who can add proteins; for some proteins they know in which order all the atoms go, but then they don't know what the thing will look like, and they have to 'solve it' to know what it looks like, and these are really important to understand their functions. There are groups that try to predict the form of proteins from energetic computations from just the formula, and they have it organized so that they have a way to objectively compare how good their predictions are with what the grand truth is by finding out about proteins that will be solved. I mean you can reasonably predict when things will get solved, but are not solved yet. So they use all those formulas, and they all work on it: they have a deadline. At some point it's clear when the thing will be solved, and they say 'now, you have to submit.' And you submit at that time, and it gets compared with the grand truth, and in that way you can score different prediction programs.

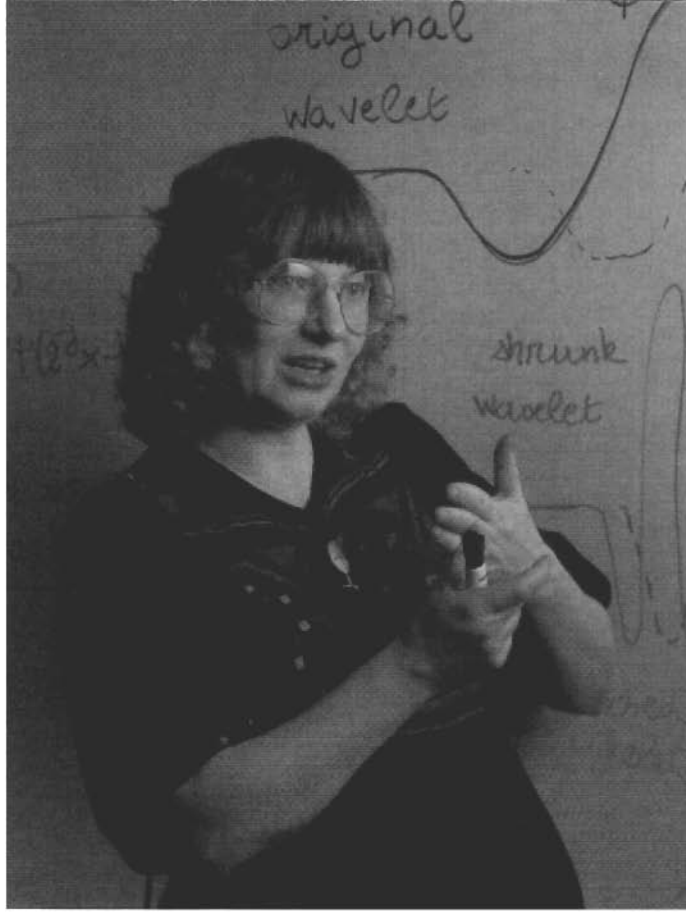
Okay, so one guy was explaining about his prediction program: because it's just too big a space to exhaustively search, he searched in a multi-resolution way. He first tried to build a coarse model, then find the best coarse model given, then build it up from there. Now his first-level coarse model, was indeed very coarse, but he was putting it on a regular lattice, and his method was doing very well. But it struck me that if we have a good idea of how to compress, how to find subdivision schemes for curves, we could look at all the proteins that they know, and try to find that subdivision scheme that will be adapted to the protein world. I've been always looking at smoothness; they don't care about smoothness. These proteins actually do all kinds of strange things. But one could try to find a subdivision scheme that would adapt to their goal, and that could give you a good idea of what kind of coarse things to start from. So that's something that hasn't started, but something I'm very excited about and I hope to work on this spring.

"Another thing that I'm very involved in is understanding the mathematical properties of coarsely quantized but very oversampled audio signals, modeled by so-called band-limited functions. There are really neat links to dynamical systems. I'd like to do that for other wavelet transforms. I think we can do it. If we can do it, I think it's going to have very useful applications, plus I think mathematically it's going to be very interesting. Already for the band-limited functions it's much more interesting that I had expected a year ago.

"I have a graduate student with whom I work on applications of wavelets to the generation and compression of surfaces. People represent surfaces with triangulations with tens and hundreds of thousands of triangles, so you'd like to compress that information. Well, you can do that via multi-resolution, and then you can wonder what kind of wavelets are associated with that. Then you can think about smoothness. In some applications, smoothness again is very important. So that's another project.

"What else? I look at my students and collaborators because everything I work on I work on with a collaborator. I'm still working with one of my former students on a way of using frames for transmitting information over multiple channels, but that's very theoretical work, and I'm not sure how close it's going to get to applications. I have the impression I'm forgetting something.

"I love to talk about mathematics and so I enjoy all the courses I teach. I teach regular undergraduate math courses, I developed a course of mathematics for non-math majors. For many students in our calculus classes this will be their last contact with mathematics. I don't think this is a very good idea. Many of them are really not turned on by calculus, and it's hard to get a really meaningful application in a course if you also want to teach calculus tools. You can try to fit in some applications, but they really feel very contrived. The real applications of calculus are all the physics and other math courses, but they won't ever see that, and they're not interested in seeing that. I wanted to get mathematical ideas across without teaching technique.



Daubechies, pregnant with daughter Carolyn, lecturing on wavelets.

"I call the course Math Alive; in two-week units we visit different concepts. I do one on voting and fair share, and one on error correction and compression, and I have one on probability and statistics, and one on cryptography, and there's one (actually not taught by me, it's a course we co-teach) on Why Newton Had to Invent Calculus, that's a unit students have more trouble with. Then there's one on dynamical systems and population explosion, it gets into population models. I've enjoyed this course a lot. I'm teaching it this spring for the fifth time, and I'd like to document it so that it can be taught by somebody else the next time, because that's the best way to prove a concept, if it can be done by somebody else.

"I want these students to go away knowing that mathematics is really important, that it turns up in lots of things where they may be impressed by the technology, but they don't realize there's very deep mathematics in there. I want them to see that mathematics is neat in that you really solve a problem. You think your way out of it. And basically, if they remember that, that's fine. I'm sure that when people meet historians, they don't say, "you must know all the dates." They know it's something else. Well, they don't know that it's something other than balancing a checkbook in mathematics. I'd like them to know that.

"I also teach graduate courses — often a starting graduate course in wavelets, sometimes a more advanced course. I enjoy teaching undergraduate courses more than graduate courses. Not because I don't like teaching graduate courses, but because at graduate study level, the starting graduate courses I can see work well, but I think an advanced graduate course works better as a reading course than as a lecture course."

An American Life

Daubechies is married to Robert Calderbank, a distinguished British mathematician. They have two children, Carolyn and Michael. If you're wondering what it must be like to have a genius for a mom, well, it sounds a lot like having a mom.

"I go to my children's school and the tables are in groups and the classroom is full of wonderful things. In my school days classrooms might have had some stuff, but we had little desks which were all lined up in rows, maybe that's the way things were here as well in the sixties. I think the new way is much more fun. It may well be different in Belgium now; I haven't visited the elementary schools. All my elementary and secondary education was in single-sex schools. That was the way it was in Belgium at that time. I went to public schools, but all public schools at that time were gender-separated.

"I have mixed feelings about that. At the time I thought it was a bad thing; it's better if you don't see another gender as a different species because at some point you will start looking for a companion, and if you haven't really met any people of the other gender until you're 18, it's very artificial. So I didn't like it at the time, but then after I got to university I realized that people in classrooms were less likely to ask me to give an answer than some boys that were there. At least in the beginning. After a while, when they knew I was interested, then it was different. But you always felt that there was a bigger hurdle to get over as a girl to get noticed than as a boy. I hadn't thought about that in great detail, but then coming to this country and hearing all the debate here about it — I can see how it might be good for some girls to have separate gender schools. In an ideal world there would not be this effect, but given it exists, I can see how it might help in building self-assurance in girls. My daughter goes to public school and it's mixed gender, and she's happy, but I'm wondering if at some point there might be a problem. I think she's a smart little girl, and if at some point I feel that because she is in a mixed-gender school she is not getting as much of an opportunity, I might consider a single-gender school. I haven't, she's only seven, and it's not an issue at this time. It's something that ten years ago I would not have thought I would ever consider, but now I would.

"I don't know if going to all-girl schools had an effect on me. My parents always made it clear I could do anything. It didn't occur to me until I went to university that people could think I was less good at something because I was a girl. I think I was very fortunate, because at that age, you're too old to take that prejudice seriously, and when you encounter someone with that attitude, you think, 'You're a jerk.'

"As I said, my parents, especially my father, really influenced my education. So did popular psychology actually. When I was little the prevailing theory was that it wasn't good to mix languages too early. You might really confuse children and then they wouldn't really be able to use any of the languages in great depth and it would leave marks on them the rest of their lives. My parents were in an ideal situation to bring my brother and I up bilingually, because they spoke French to each other and we lived in the Flemish part of Belgium, where people speak Dutch. But since there was this myth that mixing languages early was not good, they decided to bring us up in Dutch, which is my mother's tongue; my father's fluent in it since he went to school in Dutch.

"Theories having changed now; I bring my children up bilingually in Dutch and English. I thought it would be too hard doing it in a language that is not my mother tongue. My French is fluent, but in the beginning especially it took quite an effort to have a bilingual household because my husband is British and he didn't speak Dutch. He has learned together with the children. So I speak English with him, but Dutch with the children.

"My husband since he has learned Dutch would like to have practice speaking, but my children won't allow it. They roll on the floor when he tries it. They think it's quite incredibly funny. My son went through a stage where when my husband would say something in English that he knew was of interest to me too, he would turn to me and translate for me.

"English is their first language because they go to school in English. So in Dutch they sometimes have more difficulty finding words, but I really try to encourage them to find the words rather than switch to English, and so sometimes before my daughter tries to tell me something, she'll say 'how do you say that word in Dutch?' They had been talking in school about different languages, and she really wanted to tell me, so she said, in Dutch 'How do you say 'Dutch' in Dutch?' But at the end of the sentence the name was already there.

"Our son is talented in math, and I think our daughter might be too. We, of course, like to stimulate that when we ask questions, but we are not pushing them hard. What we are pushing is that they have to cooperate and work at school and do the best they can, not just in math, but in everything. In fact, I'm more concerned about writing and things like that. Trying to get ideas in an organized way on paper I think is important in mathematics as well as elsewhere.

"I like to go to my children's school and help out, especially on science day. When I'm there I'm not that woman professor mathematician, I'm Michael's mom and Carolyn's mom, and I like that. I like that." ■

Index

Title Index

Author Index

Problems and Solutions

Solutions

September 1999

November 1999

April 2000

Problems Proposed

Problems Solved

Problem Section

Guest Editors

Titu Andreescu
American Mathematics Competitions

Kiran Kedlaya
Massachusetts Institute of Technology

This section features problems for students at the undergraduate and (challenging) high school levels. Included are "Mathematical Quickies," problems which can be solved laboriously but with proper insight and knowledge can be solved easily. These problems will not be identified as such except for their solutions appearing at the end of the section (so no solutions should be submitted for these problems). "S"designated problems are problems set particularly for secondary school students and/or undergraduates.

All solutions should be submitted in duplicate in easily legible form on separate sheets containing the contributor's name, mailing address, school affiliation, and academic status (i.e., high school student, undergraduate, teacher, etc.) and sent to Steve Kennedy, Department of Mathematics, Carleton College, Northfield, MN 55057.

Proposals

To be considered for publication, solutions to the following problems should be received by September 1, 2000.

S-39. Proposed by Gregory Galperin, Eastern Illinois University. Prove, for positive real numbers $x_1, x_2, \ldots, x_n$, that

$$\left(x_1+\frac{1}{x_1}\right)\cdots\left(x_n+\frac{1}{x_n}\right)\geq\left(x_1+\frac{1}{x_2}\right)\cdots\left(x_n+\frac{1}{x_1}\right)\geq 2^n.$$

S-40. Proposed by Titu Andreescu, American Mathematics Competitions. Find all solutions to the system of equations

$$6(x-y^{-1})=3(y-z^{-1})=2(z-x^{-1})=xyz-(xyz)^{-1}$$

in nonzero real numbers x, y, z.

S-41. Proposed by Gerald Heuer, Concordia College. If a and b are rational and $a \neq 0$, the linear function $ax+b$ assumes rational values when x is rational and irrational values when x is irrational. Are there any polynomials of degree greater than 1 with this property?

S-42. Proposed by Titu Andreescu, American Mathematics Competitions. Let $a_1, \ldots, a_n$ be real numbers such that $a_1+\cdots+a_n \geq n^2$ and $a_1^2+\cdots+a_n^2 \leq n^3+1$. Prove that $n-1 \leq a_k \leq n+1$ for all k.

Problem 131. Proposed by James Propp, University of Wisconsin, Madison. The vertices of a dodecahedron are labeled with real numbers so that adjacent vertices are labeled by numbers that differ by at most 1. Prove that there exists a pair of antipodal vertices labeled by numbers that differ by at most 1.

Problem 132. Proposed by Zuming Feng, Phillips Exeter Academy. Let $ABCD$ and $EFGH$ be two squares in the plane such that they have the same center O, AB is parallel to EF, and $AB > EF$. Find all points P in the plane with the following properties.

(a) P lies outside of $ABCD$.
(b) There exists a circle centered at P that meets $ABCD$ at K and N and meets $EFGH$ at L and M, such that $KLMN$ is a convex quadrilateral and the lines KL and MN meet at O.

Problem 133. Proposed by Cecil Rousseau, University of Memphis. At a party attended by $4n$ persons, no three persons are mutually acquainted, and no three have the same number of acquaintances at the party. Prove that the set of persons attending the party can be partitioned into two subsets A and B, each consisting of mutual strangers.

Problem 134. Proposed by Titu Andreescu, American Mathematics Competitions. Let $f: [-\pi/2, \pi/2] \to [-1, 1]$ be a differentiable function whose derivative is continuous and nonnegative. Prove that there exists $x_0 \in [-\pi/2, \pi/2]$ such that

$$(f(x_0))^2+(f'(x_0))^2 \leq 1.$$

Problems S-39, S-41, 131, 132, and 133 were originally submitted for the USAMO 2000.

Solutions

S-32. A Point Not on the Euler Line

Prove that the incenter of a triangle lies on its Euler line (the line through the circumcenter and the centroid), if and only if the triangle is isosceles.

Solution by Zuming Feng, Phillips Exeter Academy. Let I, H, O be the incenter, orthocenter, circumcenter of ABC, respectively. It is a basic fact from plane geometry (obtained by chasing appropriate angles) that the lines AH and AO are interchanged by a reflection through AI. Therefore AI is an angle bisector (internal or external) of the triangle AOH, and similarly with B or C in place of A.

Now suppose I lies on the Euler line OH. If OH passes through a vertex of ABC, say A, then the line AOH is an altitude from A (since it passes through the orthocenter) and a median from A (since it passes through the centroid), and so ABC is isosceles. Suppose on the contrary that none of A, B, C lies on OH. Then by the conclusion of the previous paragraph and the angle bisector theorem,

$$OI/IH = OA/AH = OB/BH = OC/CH.$$

Since $OA = OB = OC$, we have $HA = HB = HC$ and so H is the circumcenter of O and ABC is equilateral (which gives a contradiction, since such a triangle has no Euler line).

We conclude that if I lies on the Euler line, then ABC is isosceles.

S-34. Sums of Consecutive Squares

Can the sum of the squares of 61 consecutive integers ever be a perfect square?

Solution by Mark Krusemeyer, Carleton College. This is impossible. Modulo 4, consecutive squares alternate $0, 1, 0, 1, \ldots$ so the sum of 61 consecutive squares is congruent to 30 or 31 modulo 4, and so cannot be a perfect square.

Solution by the editors. The sum of the squares of $x - 30, \ldots, x + 30$ is

$$61x^2 + 2(1^2 + \cdots + 30^2) = 61x^2 + 10 \times 31 \times 61.$$

Suppose this equals y^2. Modulo 31, we have

$$y^2 \equiv 61x^2 \equiv -x^2.$$

However, since $31 \equiv 3 \pmod 4$, -1 is not a quadratic residue modulo 31, so $y^2 \equiv -x^2 \pmod{31}$ is only possible if x and y are divisible by 31. But then $y^2 - 61x^2 = 10 \times 31 \times 61$ would be divisible by 31^2, contradiction.

Problem 123. A Harmonic Integral

Evaluate the integral $\int_0^\infty x\, d(1 - e^{-x})^n$.

Solution by the editors. Let $f(n)$ denote the given integral. We will show that $f(n) - f(n-1) = 1/n$, and notice that $f(0) = 0$ since the integrand is identically zero. From this it will follow that

$$f(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Now some calculation shows that

$$\begin{aligned} f(n) - f(n-1) &= \int_0^\infty x\, d\big[(1 - e^{-x})^n - (1 - e^{-x})^{n-1}\big] \\ &= \int_0^\infty x\, d\big[-e^{-x}(1 - e^{-x})^{n-1}\big] \\ &= \int_0^\infty xe^{-x}(1 - ne^{-x})(1 - e^{-x})^{n-2}\, dx \\ &= \frac{1}{n}(1 - e^{-x})^{n-1}\big(1 - (nx+1)e^{-x}\big)\Big|_0^\infty = \frac{1}{n}. \end{aligned}$$

Problem 124. Two Equilateral Triangles

Starting with an equilateral triangle ABC and its circumcircle, we construct an equilateral triangle $A'B'C'$ so that it circumscribes the circle and the vertex C' lies on the extended line BC. Determine the angle α between BC and $B'C'$ (here A lies above BC and C' lies below BC).

Solution by the editors. Let O be the circumcenter of ABC and D the point where BC touches the incircle of ABC. Let $r = OD$ be the inradius of ABC. Then the circumradius of ABC is $2r$ and that of $A'B'C'$ is $4r$. Therefore

$$\angle OC'D = \arcsin \frac{OD}{OC'} = \arcsin \frac{1}{4}$$

and

$$\alpha = \frac{\pi}{6} - \angle OC'D = \frac{\pi}{6} - \arcsin \frac{1}{4}.$$

Problem 125. Equal Volumes

Let $f(x)$ be a continuous increasing function on $[0, a]$ such that $f(0) = 0$. Define by R the region bounded by $f(x)$, $x = a$ and $y = 0$. Now consider the solid of revolution when R is revolved about the y-axis as a sort of dish. Determine $f(x)$ such that the volume of water the dish can hold is equal to the volume of the dish itself (for all a).

Solution by the editors. It is equivalent to ask that the volume of the dish be half that of the solid of revolution formed by the rectangle $0 \le x \le a$ and $0 \le y \le f(a)$. In equations, this condition states that

$$\int_0^a 2\pi x f(x)\,dx = \frac{1}{2}\pi a^2 f(a).$$

Applying the Second Fundamental Theorem of Calculus, we obtain that

$$2\pi a f(a) = \pi a f(a) + \frac{1}{2}\pi a^2 f'(a).$$

(The differentiability of f is also a consequence of the SFT.) This implies $(\log f(a))' = f'(a)/f(a) = 2/a$ and so $f(a) = ca^2$ for some constant $c > 0$.

Problem 126. Iterating the Sine Function

The sequence $\{\pi_n\}$ is defined by $\pi_{n+1} = \pi_n + \sin \pi_n$, $n = 1, 2, \ldots$, where π_1 is a real number. (i) If $\epsilon_n = |\pi_n - \pi| \le \sqrt{20}$, show that $\epsilon_{n+1} \le \epsilon_n^3/6$. (ii) For any given number π_1 show that $\lim_{n\to\infty} \pi_n$ exists and describe the limit.

Solution by the editors.

(i) Put $t_n = \pi_n - \pi$, so that $t_{n+1} = t_n - \sin t_n$. Then

$$t_{n+1} = \frac{t_n^3}{3!} - \sum_{i=1}^{\infty} \left(\frac{t_n^{4i+1}}{(4i+1)!} - \frac{t_n^{4i+3}}{(4i+3)!} \right).$$

Assuming $0 \le t_n \le \sqrt{20}$, all of the terms in the sum are positive (since $t_n^2 \le 20 \le (4i+2)(4i+3)$), so $|t_{n+1}| \le |t_n^3/6|$. The argument is similar for $0 \ge t_n \ge -\sqrt{20}$.

(ii) We will show that

$$\lim_{n\to\infty} \pi_n = \begin{cases} 2m\pi & \pi_1 = 2m\pi \\ (2m+1)\pi & 2m\pi < \pi_1 < (2m+2)\pi. \end{cases}$$

Let us first consider the case $0 < \pi_1 < \pi$. As in (i), put $t_n = \pi_n - \pi$, so that $0 < t_1 < \pi$ and the claim is that $\lim_{n\to\infty} t_n = 0$. Given that $0 < t_n < \pi$, we have $0 < t_{n+1}$ since $\sin x < x$ for all $x > 0$, and $t_{n+1} < t_n$ since $\sin t_n > 0$. Thus the t_n, forming a decreasing sequence of positive real numbers, converge to a limit $L \in [0, t_1]$. However,

$$L = \lim_{n\to\infty} t_n = \lim_{n\to\infty} t_{n-1} - \sin t_{n-1} = L - \sin L$$

by continuity. The only possible $L \in [0, t_1]$ is thus $L = 0$, and so $\lim_{n\to\infty} t_n = \pi$.

If $\pi < \pi_1 < 2\pi$, we can apply the above reasoning to the sequence starting with $2\pi - \pi_1$, since $(2\pi - x) + \sin(2\pi - x) = 2\pi - (x + \sin x)$, and deduce that $\lim_{n\to\infty} \pi_n = \pi$ again. For the general case, note that $(2\pi n + x) + \sin(2\pi n + x) = 2\pi n + (x + \sin x)$, so we can deduce the general case by choosing n so that $2\pi n + \pi_1 \in (0, 2\pi)$ and replacing π_1 with $2\pi n + \pi_1$.

S-39. (Quickie) Two Inequalities

To prove the left inequality, multiply through by $x_1 \ldots x_n$, so that the desired result becomes

$$(x_1^2 + 1) \cdots (x_n^2 + 1) \ge (x_1 x_2 + 1) \cdots (x_n x_1 + 1).$$

Then note that $(x_1^2 + 1)(x_2^2 + 1) \ge (x_1 x_2 + 1)^2$, since the difference between the two sides is $x_1^2 + x_2^2 - 2x_1 x_2$. Multiplying the analogous inequalities and taking square roots yields the claim.

To prove the right inequality, note that $x_1 + 1/x_2 \ge 2\sqrt{x_1/x_2}$ by AM-GM. Multiplying the analogous inequalities yields the claim.

S-40. (Quickie) A System of Equations

The only solutions are $x = y = z = 1$ and $x = y = z = -1$. The given equalities imply that

$$(x - y^{-1}) + (y - z^{-1}) + (z - x^{-1}) = xyz - (xyz)^{-1},$$

which factors as

$$(x - y^{-1})(y - z^{-1})(z - x^{-1}) = 0.$$

Thus one of $x - y^{-1}, y - z^{-1}, z - x^{-1}$ is zero, but the given equalities then imply that all three are zero. Thus $xy = yz = zx = 1$, $(xyz)^2 = 1$ and so $x = y = z = 1$ or $x = y = z = -1$.

Illustration by John Johnson.

Do You Haiku?

Those among us who fancy ourselves amateur wordsmiths or late-night poets — or who just need a distraction from our Calculus takehome — have penned our fair share of limericks and haiku. Occasionally our burning passion for mathematics drives our Muse to whisper beautiful (and sometimes even rhyming) mathematics in our ear as we write. After all, who among us has not, on considering Euclid's *Elements*, felt the creative inspiration to write "There once was an old man from Crete, Whose parallel lines did not meet,..."

Perhaps your brain prefers the opposite metric (long, short, short, long, short, short), found in the double-dactyl:

> Mathematicians are
> Totally awesome, they
> Teach us mathematics,
> Then leave with a smirk
>
> Homework assigned gives us
> Nightmares at bedtime with
> Epsilon-delta proofs
> Which will not work.

Enter the contest—
Pour thy soul onto paper.
Win a free t-shirt.

Summon up your creative juices, sharpen your pencil and grab some paper for this issue's contest.

Two years ago, *Salon Magazine* suggested to readers to create computer error messages in haiku. (Haiku is an epigrammatic Japanese poetry form of three unrhyming lines with syllable counts 5, 7, 5.) Many of the entries were quite clever:

> With searching comes loss
> and the presence of absence:
> "My Novel" not found.
> — Howard Korder
>
> A crash reduces
> your expensive computer
> to a simple stone.
> — James Lopez
>
> Having been erased,
> The document you're seeking
> Must now be retyped.
> — Judy Birmingham
>
> Out of memory.
> We wish to hold the whole sky,
> But we never will.
> — Francis Heaney
>
> A file that big?
> It might be very useful.
> But now it is gone.
> — David J. Liszewski

Here is the chance of a lifetime to become a published poet. Write some mathematical haiku: pithy or profound, witty or sober, simple or eloquent, and send it to **dhaunspe@carleton.edu** by May 15th. Selected entries will be published in the next *Math Horizons*; the best entries will receive *Math Horizons* t-shirts. This is your opportunity to bare your mathematical soul before all humankind — and to put off your takehome just a few minutes more.