Mechanics
Rotational Kinematics

Lana Sheridan

De Anza College

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Last time

- ballistic pendulum
- 2-D collisions (not on test)
- rotational quantities

Overview

- relation of angular to translational quantities
- rotational kinematics
- torque (?)

Angular Quantities

change in angle, $\Delta\theta$ (rad)

angular speed, $\omega = \frac{d\theta}{dt}$ (rad/s)

angular acceleration, $\alpha = \frac{d\omega}{dt}$ (rad/s^2)

Rotation of Rigid Objects and Vector Quantities

We can also define these quantities as vectors! (Provided we fix the axis of rotation.)

Rotation of Rigid Objects and Vector Quantities

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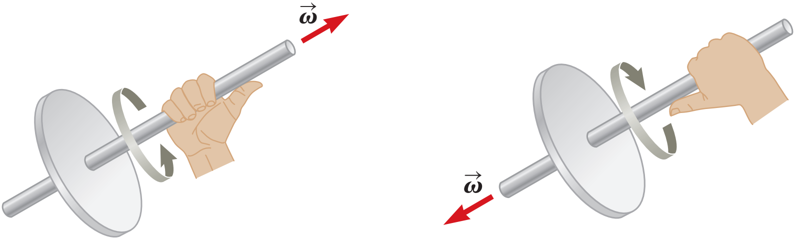
The angle can be positive or negative depending on whether it is clockwise or counterclockwise from the reference point.

This is the same way that a 1-dimensional displacement x can be positive or negative based on whether it is on the left or right of the origin.

Rotation of Rigid Objects and Vector Quantities

By convention, we define the counterclockwise direction to be positive. The vector itself is drawn along the axis of rotation.



Then we can write:

$$\boldsymbol{\theta} = \frac{s}{r}\,\hat{\mathbf{n}} \quad ; \quad \boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt} \quad ; \quad \boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of rotation.

Comparison of Linear and Rotational quantities

Linear Quantities	Rotational Quantities
$\mathbf{x}$	θ
$\mathbf{v} = \frac{d\mathbf{x}}{dt}$	$\boldsymbol{\omega} = \frac{d\theta}{dt}$
$\mathbf{a} = \frac{d\mathbf{v}}{dt}$	$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$

Rotational Kinematics

If α is constant, we have basically the same kinematics equations as before, but the relations are between the new quantities.

$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha \cdot \Delta\theta$$

$$\omega_{\text{avg}} = \frac{1}{2}(\omega_i + \omega_f)$$

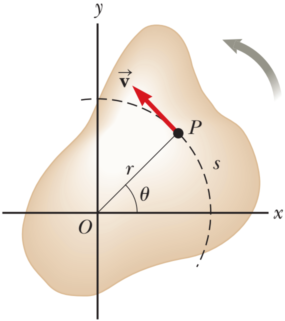
$$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$$

Kinematics Comparison

Linear Quantities	Rotational Quantities
$\mathbf{v}_f = \mathbf{v_i} + \mathbf{a}t$	$\boldsymbol{\omega}_f = \boldsymbol{\omega}_i + \boldsymbol{\alpha}t$
$\mathbf{x}_f = \mathbf{x_i} + \mathbf{v_i}t + \frac{1}{2}\mathbf{a}t^2$	$\theta_f = \theta_i + \boldsymbol{\omega}_i t + \frac{1}{2}\boldsymbol{\alpha}t^2$
$v_f^2 = v_i^2 + 2\mathbf{a} \cdot \boldsymbol{\Delta x}$	$\omega_f^2 = \omega_i^2 + 2\boldsymbol{\alpha} \cdot \boldsymbol{\Delta\theta}$
$\mathbf{v}_{avg} = \frac{1}{2}(\mathbf{v_i} + \mathbf{v_f})$	$\boldsymbol{\omega}_{\mathrm{avg}} = \frac{1}{2}(\boldsymbol{\omega}_i + \boldsymbol{\omega}_f)$
$\mathbf{x_f} = \mathbf{x_i} + \frac{1}{2}(\mathbf{v_i} + \mathbf{v_f})t$	$\theta_f = \theta_i + \frac{1}{2}(\boldsymbol{\omega}_i + \boldsymbol{\omega}_f)t$

Relating Rotational Quantities to Translation of Points

Consider a point on the rotating object. How does its speed relate to the angular speed?



We know $s = r\theta$, so since the object's speed is its speed along the path s,

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Relating Rotational Quantities to Translation of Points

Since $\omega = \frac{d\theta}{dt}$, that gives us and expression for (tangential) speed

$$v = r\omega$$

And differentiating both sides with respect to t again:

$$a_t = r\alpha$$

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Notice that the above equation gives the rate of change of speed, which is the *tangential acceleration*.

Centripetal Acceleration

Remember:

$$a_t = \frac{\mathrm{dv}}{\mathrm{dt}}$$

where v is the speed, not velocity.

So,

$$a_t = r\alpha$$

But of course, in order for a mass at that point, radius r, to continue moving in a circle, there must be a centripetal component of acceleration also.

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For a rigid object, the force that supplies this acceleration will be some internal forces between the mass at the rotating point and the other masses in the object. Those are the forces that hold the object together.

Example 1

Page 325, #5

$\boxed{\textbf{5.}}$ **W** A wheel starts from rest and rotates with constant angular acceleration to reach an angular speed of 12.0 rad/s in 3.00 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time interval.

Example 1

Known: ω_i, ω_f, t

(a) Angular accel., α?

$$\begin{aligned}
\omega_f &= \omega_i + \alpha t \\
\alpha &= \frac{\omega_f - \omega_i}{t} \\
\alpha &= \frac{12.0 \text{ rad/s} - 0}{3.00 \text{ s}} \\
\alpha &= 4.00 \text{ rad s}^{-2}
\end{aligned}$$

Example 1

Known: ω_i, ω_f, t, α

(b) Angle in radians, $\Delta\theta$?

Either use

$$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}(4)(3)^2 = 18 \text{ radians}$$

or use

$$\begin{aligned}
\Delta\theta &= \frac{1}{2}(\omega_i + \omega_f)t \\
\Delta\theta &= \frac{1}{2}(0 + 12.0)(3.00) \\
&= 18.0 \text{ radians}
\end{aligned}$$

Example 2

Page 325, #8

8. A machine part rotates at an angular speed of 0.060 rad/s; its speed is then increased to 2.2 rad/s at an angular acceleration of 0.70 rad/s^2. (a) Find the angle through which the part rotates before reaching this final speed. (b) If both the initial and final angular speeds are doubled and the angular acceleration remains the same, by what factor is the angular displacement changed? Why?

Example 2

Known: ω_i, ω_f, α

(a) $\Delta\theta$?

$$\begin{aligned}
\omega_f^2 &= \omega_i^2 + 2\alpha\Delta\theta \\
\Delta\theta &= \frac{\omega_f^2 - \omega_i^2}{2\alpha} \\
&= \frac{(2.2)^2 - (0.060)^2}{2 \times 0.70} \\
&= 3.5 \text{ rad}
\end{aligned}$$

Example 2

(b) If both ω_i and ω_f are doubled, α kept constant, what happens to $\Delta\theta$?

$$\begin{aligned}
\Delta\theta' &= \frac{(2\times\omega_f)^2-(2\times\omega_i)^2}{2\alpha} \\
&= 4\,\frac{\omega_f^2-\omega_i^2}{2\alpha} \\
&= 4\,\Delta\theta
\end{aligned}$$

Torque

Torque is a measure of force-causing-rotation.

It is not a force, but it is related. It depends on a force vector and its point of application relative to an axis of rotation.

Torque is given by:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

That is: the cross product between

- a vector $\mathbf{r}$, the displacement of the point of application of the force from the axis of rotation, and
- an the force vector $\mathbf{F}$

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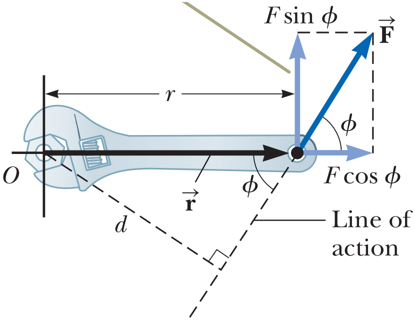
Units: N m   Newton-meters. These are not Joules!

Torque



$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = rF \sin \phi \, \hat{\mathbf{n}}$$

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{F}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to $\mathbf{r}$ and $\mathbf{F}$, as determined by the right-hand rule.

Vectors Properties and Operations

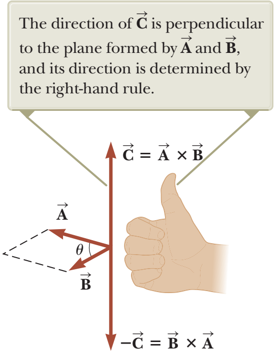
Multiplication by a vector:

The Cross Product

Let $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$
$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j}$,

$$\mathbf{A} \times \mathbf{B} = (A_x B_y - A_y B_x)\mathbf{k}$$

The output of this operation is a vector.



Equivalently,

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \ \hat{\mathbf{n}}_{\mathbf{AB}}$$

where $\hat{\mathbf{n}}_{\mathbf{AB}}$ is a unit vector perpendicular to $\mathbf{A}$ and $\mathbf{B}$.

Vectors Properties and Operations

(See page 336 in Serway and Jewett.)

The Cross Product - with k components

In general:
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k},$$

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

How do we usually implement this formula?
Via the determinant of a matrix:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector Operations: Cross Product Practice

Try it yourself! Find $\mathbf{A} \times \mathbf{B}$ when:

$$\mathbf{A} = 1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\ ; \quad \mathbf{B} = -1\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$$

Vector Operations: Cross Product Practice

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Now find $\mathbf{B} \times \mathbf{A}$...

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Now find $\mathbf{B} \times \mathbf{A}$...
First $\mathbf{A} \times \mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = 22\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$$

Vector Operations: Cross Product Practice

Try it yourself! Find $\mathbf{A} \times \mathbf{B}$ when:

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First $\mathbf{A} \times \mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = 22\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{B} \times \mathbf{A} = -22\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$$

Vectors Properties and Operations

(See page 336 in Serway and Jewett.)

The Cross Product - with k components

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \ \hat{\mathbf{n}}_{\mathbf{AB}}$$

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The Cross Product - with k components

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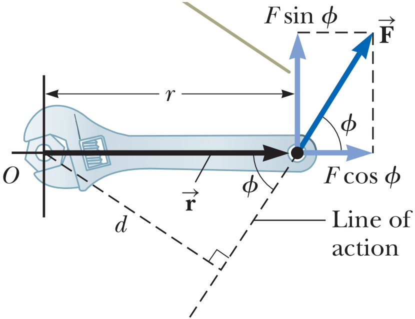
Properties

- The cross product is **not** commutative: $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$.
  In fact, it is *anticommutative* because $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$.
- If $\mathbf{A} \parallel \mathbf{B}$, $\mathbf{A} \times \mathbf{B} = 0$.
- If $\mathbf{A} \perp \mathbf{B}$, $\mathbf{A} \times \mathbf{B} = AB \ \hat{\mathbf{n}}_{\mathbf{AB}}$.

Torque



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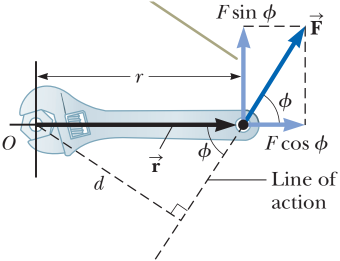
where ϕ is the angle between $\mathbf{r}$ and $\mathbf{F}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to $\mathbf{r}$ and $\mathbf{F}$, as determined by the right-hand rule.

Torque

Diagram also illustrates two points of view about torque:



$$\boldsymbol{\tau} = r(F \sin \phi)\, \hat{\mathbf{n}}$$

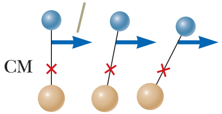
or

$$\boldsymbol{\tau} = F(r \sin \phi)\, \hat{\mathbf{n}}$$

In the diagram, the distance $d = r \sin \phi$ and is called the “moment arm” or “lever arm” of the torque.

Torque

Torque:



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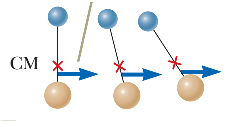
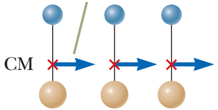
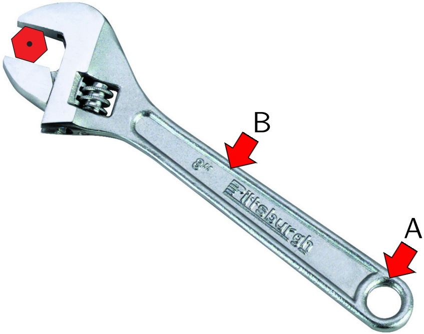


No torque:



Question



A torque is supplied by applying a force at point A. To produce the same torque, the force applied at point B must be:

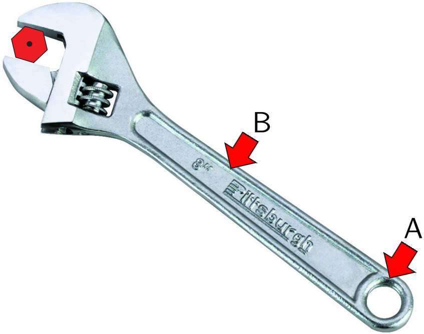
(A) greater

(B) less

(C) the same

[1] Image from Harbor Freight Tools, www.harborfreight.com

Question



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Summary

- relation of angular to translational quantities
- rotational kinematics
- torque (?)

2nd Test tomorrow.

Homework

- set yesterday: Ch 10 Probs: 2, 7, 9, 13, 19, 29 (won't be on 2nd test)