# Mechanics
# Rotational Inertia

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# Last time

- torque
- net torque
- static equilibrium

# Overview

- another static equilibrium example
- Newton's second law for rotation
- moment of inertia
- (parallel axis theorem)

# Question

**Quick Quiz 12.3**[1] A meterstick of uniform density is hung from a string tied at the 25-cm mark. A 0.50-kg object is hung from the zero end of the meterstick, and the meterstick is balanced horizontally. What is the mass of the meterstick?

**(A)** 0.25 kg

**(B)** 0.50 kg

**(C)** 1.0 kg

**(D)** 2.0 kg

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[1] Serway & Jewett, page 366.

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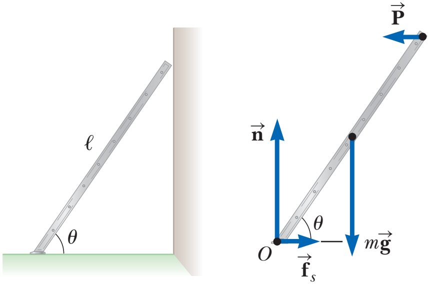
# Example - Slipping Ladder

A uniform ladder of length $\ell$, rests against a smooth, vertical wall. The mass of the ladder is $m$, and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$. Find the minimum angle $\theta_{min}$ at which the ladder does not slip.



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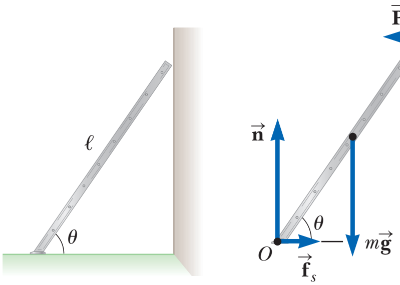



$$\theta_{min} = 51^\circ$$

# Rotational Version of Newton's Second Law

Tangential components of forces give rise to torques.

They also cause tangential accelerations. Consider the tangential component of the net force, $\mathbf{F}_{\text{net},t}$:

$$\mathbf{F}_{\text{net},t} = m\, a_t$$

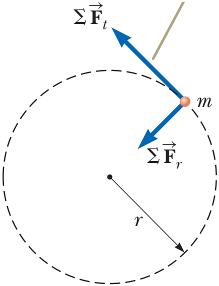
from Newton's second law.

$$\boldsymbol{\tau}_{\text{net}} = \mathbf{r} \times \mathbf{F}_{\text{net}} = r\, F_{\text{net},t}\, \hat{\mathbf{n}}$$

Now let's specifically consider the case of a single particle, mass $m$, at a fixed radius $r$.

# Rotational Version of Newton's Second Law

A single particle, mass $m$, at a fixed radius $r$.




For such a particle, $\mathbf{F}_{\text{net},t} = m\mathbf{a}_t$

$$\begin{aligned}\boldsymbol{\tau}_{\text{net}} &= rF_{\text{net},t}\,\hat{\mathbf{n}} \\ &= r\,m\,a_t\,\hat{\mathbf{n}} \\ &= r\,m\,(\boldsymbol{\alpha} r) \\ &= (mr^2)\,\boldsymbol{\alpha}\end{aligned}$$

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Replacing the constant quantity in our expression for $\boldsymbol{\tau}_{\text{net}}$:

$$\boldsymbol{\tau}_{\text{net}} = I\boldsymbol{\alpha}$$

# Rotational Version of Newton's Second Law

Compare!

$$\boldsymbol{\tau}_{\text{net}} = I\boldsymbol{\alpha}$$

$$\mathbf{F}_{\text{net}} = m\mathbf{a}$$

Now the moment of inertia, $I$, stands in for the inertial mass, $m$.

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The moment of inertia measures the rotational inertia of an object, just as mass is a measure of inertia.

# Rotational Inertia, or, Moment of Inertia

We just found that for a single particle, mass $m$, radius $r$,

$$I = mr^2$$

However, this will not be the moment of inertia for an extended object with mass distributed over varying distances from the rotational axis.

For that case, the torque on each individual mass $m_i$ will be:

$$\boldsymbol{\tau}_i = m_i r_i^2 \boldsymbol{\alpha}$$

And we sum over these torques to get the net torque. So, for a collection of particles, masses $m_i$ at radiuses $r_i$:

$$I = \sum_i m_i r_i^2$$

# Moment of Inertia

Important caveat: Moment of inertia depends on the object's mass, shape, and the axis of rotation.

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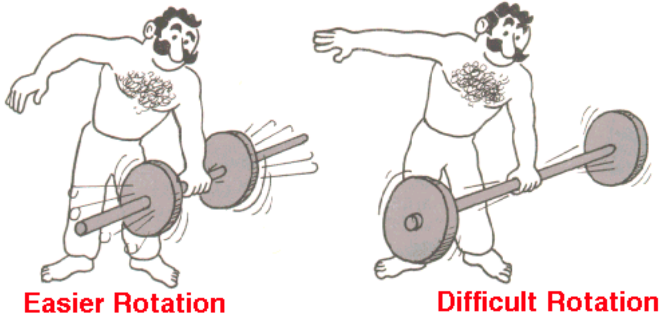
Also notice that this sum is similar to the expression for the center-of-mass, but for $I$ we have $r^2$ dependance and we do not divide by the total mass.

Units: kg $\text{m}^2$

# Moment of Inertia

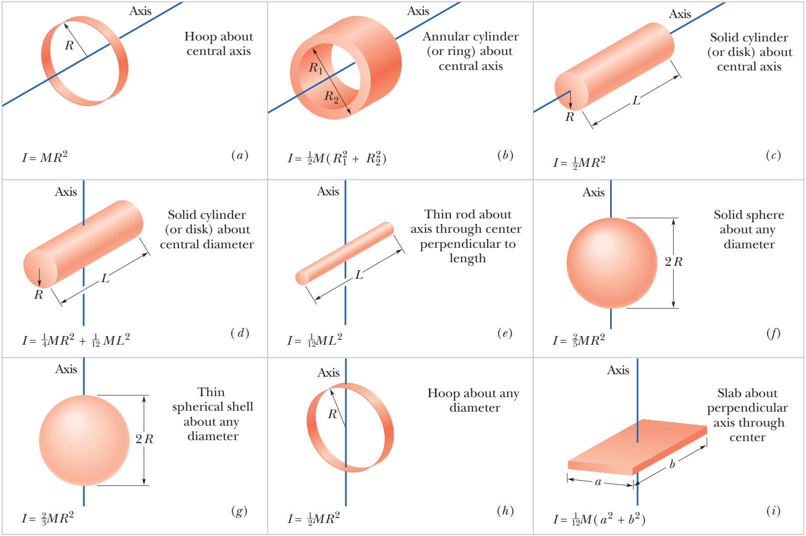
If the object's mass is far from the point of rotation, more torque is needed to rotate the object (with some angular acceleration).




The barbell on the right has a greater moment of inertia.

[1]Diagram from Dr. Hunter's page at http://biomech.byu.edu (by Hewitt?)

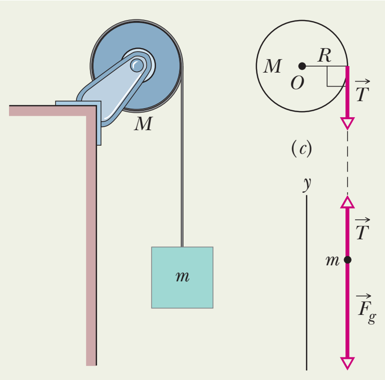
# Moments of Inertia




[1] Figure from Halliday, Resnick, Walker, 9th ed, page 255.

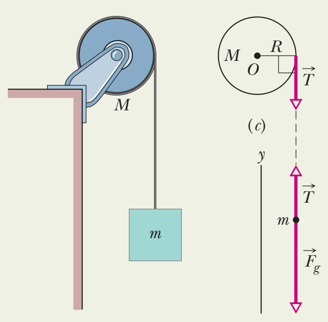
# Newton's Second Law for Rotation - Example

A block of mass $m$ is attached to a string wrapped around a pulley wheel of radius $R$ and mass $M$. The block is released from rest and descends. The pulley wheel is a uniform disk, it's axle is frictionless, and the string does not slip.




Find the acceleration of the block, $\mathbf{a}$, the angular acceleration of the pulley wheel, $\boldsymbol{\alpha}$, and the tension in the string, $T$.

# Newton's Second Law for Rotation - Example




block:

$$F_{\text{net},y} = ma_y$$
$$T - mg = -ma \qquad (1)$$

pulley:

$$\tau_{\text{net}} = I\alpha$$
$$RT = \frac{1}{2}MR^2\alpha$$
$$T = \frac{1}{2}MR\alpha \qquad (2)$$

rotational and translational acceleration:

$$a = R\alpha \qquad (3)$$

# Newton's Second Law for Rotation - Example

(3) into (2):

$$T = \frac{1}{2}Ma$$

Putting that into (1) and rearranging:

$$\begin{aligned}\left(\frac{1}{2}Ma\right) - mg &= -ma \\ \mathbf{a} &= \underline{\frac{2mg}{M+2m} \text{ downward}}\end{aligned}$$

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Eq for $a$ into (2):

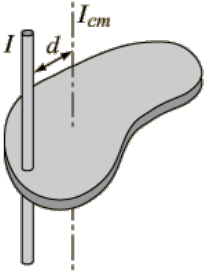
$$T = \underline{\frac{Mmg}{M+2m}}$$

Eq for $a$ into (3):

$$\boldsymbol{\alpha} = \underline{\frac{2mg}{R(M+2m)} \text{ clockwise}}$$

# Moment of Inertia - Parallel Axis Theorem

Suppose you need the moment of inertia about an axis not through the center of mass, but all you know is $I_{\text{CM}}$.



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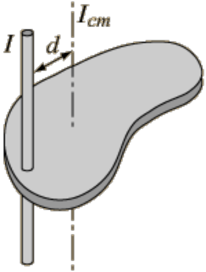
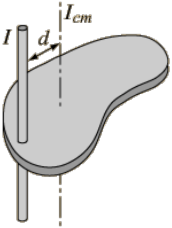



We can determine the moment of inertia about any parallel axis with a simple calculation!

$$I_{\parallel} = I_{\mathrm{CM}} + M\,d^2$$

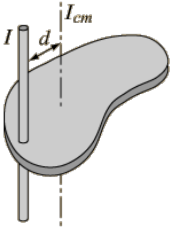
# Moment of Inertia - Parallel Axis Theorem




For an axis through the center of mass and any parallel axis through some other point:

$$I_{\parallel} = I_{\mathrm{CM}} + M\,d^2$$

# Moment of Inertia - Parallel Axis Theorem



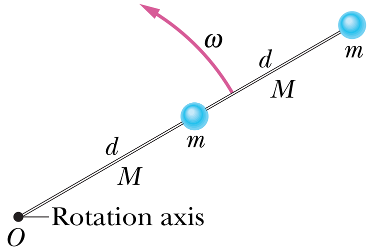

For an axis through the center of mass and any parallel axis through some other point:

$$I_{\parallel} \;=\; I_{\text{CM}} + M\,d^2$$

Use: perhaps you know $I_{\text{CM}}$, but need the moment of inertia about a different point; or maybe you know the moment of inertia about one axis, you can find $I_{\text{CM}}$, then you can find $I$ for **any** parallel axis.

# Example based on problem 41, pg 269

Two particles, each with mass $m = 0.85$ kg, are fastened to each other, and to a rotation axis at $O$, by two thin rods, each with length $d = 5.6$ cm and mass $M = 1.2$ kg. Measured about $O$, what is the combination's rotational inertia?



# Example based on problem 41, pg 269

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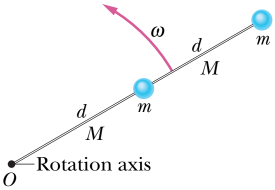


For the two particles $I = mr^2$, so in total:

$$I_{\text{particles}} = md^2 + m(2d)^2$$

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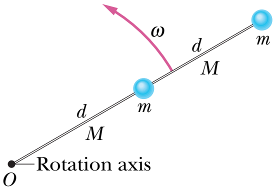


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For the rods, treat them as one, $I_{\text{CM}} = \frac{1}{12} M_{\text{tot}} L^2$, if $L = 2d$ is the rod length and $M_{\text{tot}} = 2M$.

Using the parallel axis theorem, we can find the rotational inertia through its end point, a distance $d$ away:

$$I_{\text{rod}} = \frac{1}{12}(2M)(2d)^2 + (2M)d^2$$

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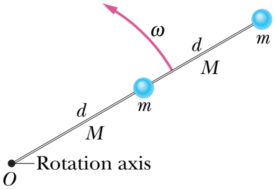


In total:

$$\begin{aligned} I_O &= md^2 + m(2d)^2 + \frac{1}{12}(2M)(2d)^2 + (2M)d^2 \\ &= 5md^2 + \frac{8}{3}Md^2 \\ &= \underline{2.3 \times 10^{-2} \text{ kg m}^2} \end{aligned}$$

# Summary

- static equilibrium example
- Newton'sk second law for rotation
- moment of inertia
- (parallel axis theorem)

## Homework

- Ch 10 Probs: 49, 53 - Newton's 2nd law for rotation
- Ch 10 Probs: 37, 43, 104(a) only [not (b)] (can wait to do)