Mechanics
Kinetic Energy of Rotation
Angular Momentum

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Last time

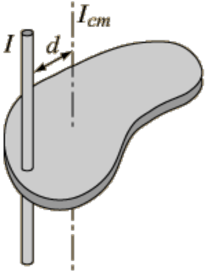
- finished static equilibrium
- Newton's second law for rotations
- rotational inertia

Overview

- parallel axis theorem for rotational inertia
- rotational kinetic energy
- angular momentum

Moment of Inertia - Parallel Axis Theorem

Suppose you need the moment of inertia about an axis not through the center of mass, but all you know is I_{CM}.



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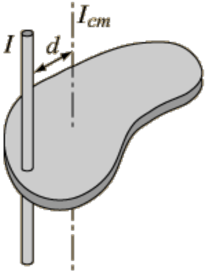
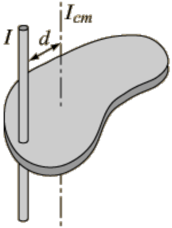


We can determine the moment of inertia about any parallel axis with a simple calculation!

$$I_{\parallel} \quad = \quad I_{\mathrm{CM}} + M\,d^2$$

Moment of Inertia - Parallel Axis Theorem



For an axis through the center of mass and any parallel axis through some other point:

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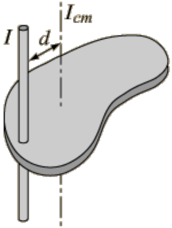


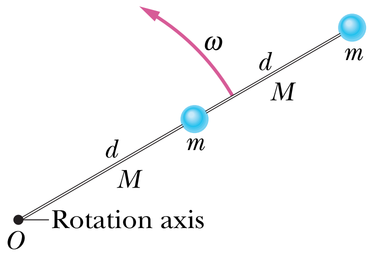
For an axis through the center of mass and any parallel axis through some other point:

$$I_{\parallel} = I_{CM} + M d^2$$

Use: perhaps you know I_{CM}, but need the moment of inertia about a different point; or maybe you know the moment of inertia about one axis, you can find I_{CM}, then you can find I for **any** parallel axis.

Example based on problem 41, pg 269

Two particles, each with mass $m = 0.85$ kg, are fastened to each other, and to a rotation axis at O, by two thin rods, each with length $d = 5.6$ cm and mass $M = 1.2$ kg. Measured about O, what is the combination's rotational inertia?



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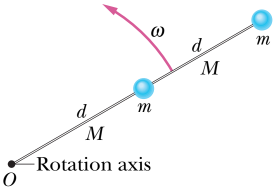

For the two particles $I = mr^2$, so in total:

$$I_{\text{particles}} = md^2 + m(2d)^2$$

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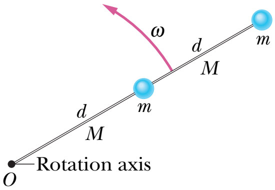

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For the rods, treat them as one, $I_{\text{CM}} = \frac{1}{12}M_{\text{tot}}L^2$, if $L = 2d$ is the rod length and $M_{\text{tot}} = 2M$.

Using the parallel axis theorem, we can find the rotational inertia through its end point, a distance d away:

$$I_{\text{rod}} = \frac{1}{12}(2M)(2d)^2 + (2M)d^2$$

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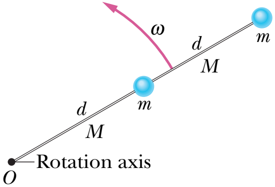

In total:

$$
\begin{aligned}
I_O &= md^2 + m(2d)^2 + \frac{1}{12}(2M)(2d)^2 + (2M)d^2 \\
&= 5md^2 + \frac{8}{3}Md^2 \\
&= \underline{2.3 \times 10^{-2}\ \text{kg m}^2}
\end{aligned}
$$

Rotational Kinetic Energy

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Imagine an object made up of a collection of particles, mass m_i, radius r_i. The kinetic energy or each particle is

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Notice that $I = \sum_i m_i r_i^2$.

Rotational Kinetic Energy

Kinetic energy of a rigid object rotating at an angular speed ω is

$$K = \frac{1}{2} I \omega^2$$

Kinetic Energy of Rotation

Quick Quiz 10.6[1] A section of hollow pipe and a solid cylinder have the same radius, mass, and length. They both rotate about their long central axes with the same angular speed. Which object has the higher rotational kinetic energy?

(A) The hollow pipe does.

(B) The solid cylinder does.

(C) They have the same rotational kinetic energy.

(D) It is impossible to determine.

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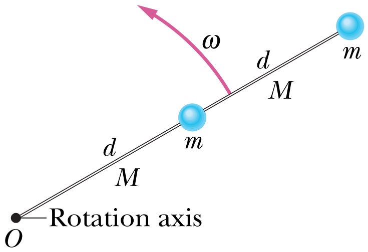
Another Example based on problem 41, pg 269

Two particles, each with mass $m = 0.85$ kg, are fastened to each other, and to a rotation axis at O, by two thin rods, each with length $d = 5.6$ cm and mass $M = 1.2$ kg. The combination rotates around the rotation axis with the angular speed $\omega = 0.30$ rad/s. What is the combination's kinetic energy?



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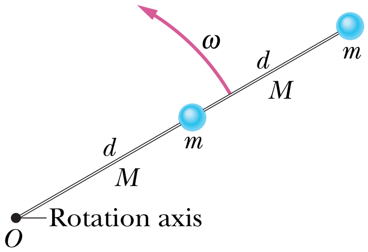

(We already found $I_{O,\text{ total}} = 2.3 \times 10^{-2}$ kg m^2.)

$$K = 1.0 \times 10^{-3} \text{ J}$$

Momentum and Rotating Objects

Imagine an solid object rotates about it's center of mass, but it's CM remains fixed.

Then $v_{\mathrm{CM}} = 0$.

Linear momentum, $\mathbf{p} = m\mathbf{v}$.

The CM is fixed, but other points on the object move. Can we assign a momentum to the object?

Reminder about Force and Torque

Torque relates to force:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

We can write Newton's Second Law in its more general form:

$$\mathbf{F}_{\text{net}} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

This relates **force** to **momentum**.

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Relating **torque** to $\mathbf{r} \times \mathbf{p}$?

Yes!

Angular Momentum

A new quantity, angular momentum:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where

- $\mathbf{r}$ is the displacement vector of a particle relative to some axis of rotation, and
- $\mathbf{p}$ is the momentum of the particle

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Units: kg m^2 s^{-1}

Angular Momentum

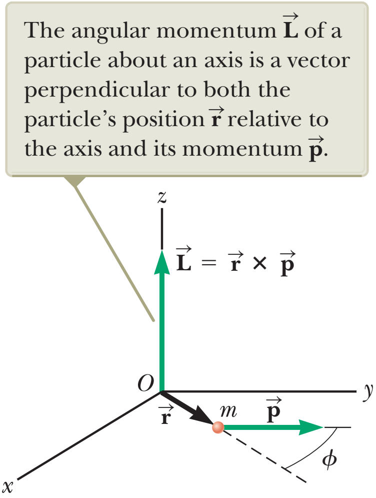
This is a more general form of Newton's second law for rotations!

$$\boldsymbol{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt}$$

$\Rightarrow$ torques cause changes in angular momentum, $\mathbf{L}$.

Angular Momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$



Angular Momentum

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is a vector equation.

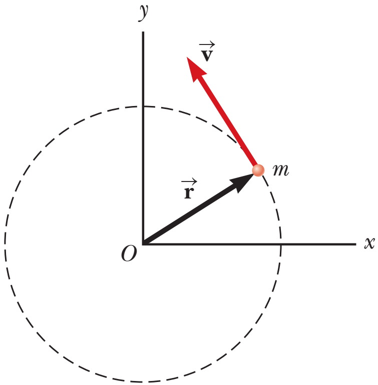
If we only need to know the magnitude of L, then we can use the following expression:

$$L = mvr \sin \phi$$

where we used $p = mv$, and ϕ is the angle between $\mathbf{r}$ and $\mathbf{p}$.

Angular Momentum of a Particle in Circular Motion

A particle has mass, m, velocity v, and travels in a circular path of radius r about a point O. What is the magnitude of its angular momentum relative to the axis O?



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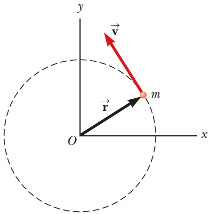

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(B) $-\mathbf{k}$

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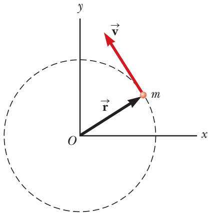

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Angular mtm of an object moving in a Straight Line

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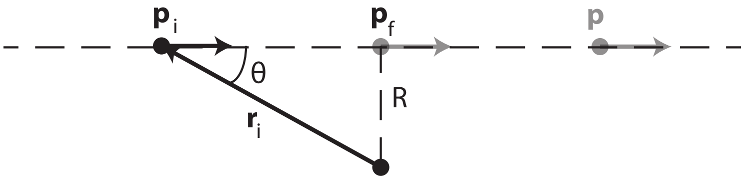
Suppose a particle of mass m passes by a point O with a distance of closest approach of R, traveling at a speed v.



What is the angular momentum of the particle about the axis O?

Isolated Object moving in a Straight Line

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But, $r_i\sin\theta_i = R$, so

$$\mathbf{L}_i = mvR\,(-\mathbf{k})$$

and $\mathbf{L}_i = \mathbf{L}_f$.

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Angular mtm of an object moving in a Straight Line

The magnitude of the angular momentum of an object, of mass m and velocity v, traveling in a straight line about an axis O is

$$L = mvR$$

where R is the **distance of closest approach** of the axis O.

Summary

- parallel axis theorem for rotational inertia
- rotational kinetic energy
- angular momentum

Homework

- Ch 10 Probs: 37, 43, 104(a) only [not (b)] (from yesterday)
- Ch 10 Probs: 33, 35
- Ch 11 Prob: 27