Lab Skills: Analyzing Errors, Significant Figures, and Measurement Uncertainties*

1 Accuracy and Precision

In practice, no measurement procedure is ever perfect. When making a series of measurements of the same quantity, each individual measurement will not necessarily return the exact value of the quantity. We can think of how useful a measurement device of procedure is by considering two features: the precision of the device and the accuracy of the device. These are two separate concepts. A very precise device may nevertheless not be accurate, and vice versa.

Precision. *A measurement is precise if it yields very similar results when repeated.*

Accuracy. *A measurement is accurate if it its result is very close to the true value.*

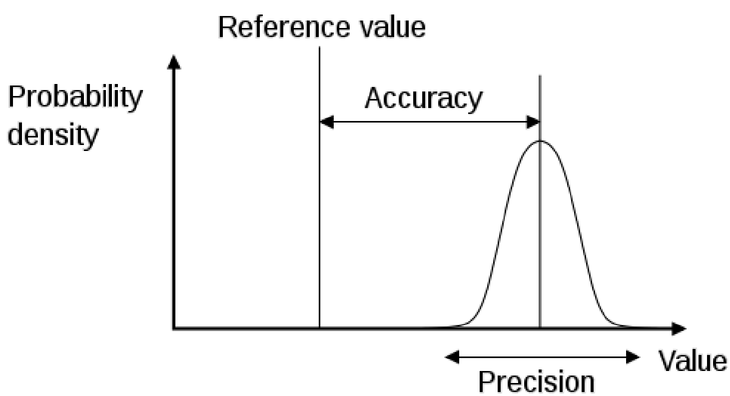

Figure 1: The magnitude of the variation of the measurement samples reflects the precision of the measurement and the distance of the average of the values from the true value reflects the accuracy. In the figure, take the marked reference value to be the true value.[2]

We can quantify the accuracy of a measurement if we know what to expect for the true value. We can do this by finding the average of the measurement samples. Suppose a quantity x is measured n times. The first measurement of x yields the value x_1, the second, x_2, the i^{th}, x_i. Then the average is given by:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

*These notes are based heavily on the notes of Prof Eduardo Luna.

[2]Graph created by Pekaje, based on PNG version by Anthony Cutler, Wikipedia.

Take the average to be the best estimate of the value. We can then find the percentage error between the average and the known true value:

$$\text{percentage error} = \frac{|\text{average} - \text{true value}|}{\text{true value}} \times 100\%$$

One way to put a value to the precision of a measurement is to find the standard deviation of the measurement samples. The standard deviation is given by[3]:

$$\sigma_x = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

If a measurement is very precise, then it will highly repeatable, in other words, not vary much as many measurements are taken. All the measurement values x_i will be close to the mean $\bar{x}$. Therefore, for a precise measurement, the standard deviation, σ_x will be very small compared to the value of $\bar{x}$. We can define the fractional uncertainty:

$$\text{fractional uncertainty} = \frac{\sigma_x}{\bar{x}}$$

A precise measurement will have a low fractional uncertainty.

A good measurement will have high precision (small fractional uncertainty) and high accuracy (small percentage difference).

2 Types of Measurement Errors

When making measurements and gathering data in experiments there will always be some uncertainty in the measured values. We refer to the uncertainty as the error in the measurement. These errors can result in data that is not accurate, or lead to variation between different runs of an experiment. Errors fall into two categories:

1. **Systematic Errors** - errors resulting from measuring devices being out of calibration. Such measurements will be consistently too small or too large. These errors can be eliminated by pre-calibrating against a known, trusted standard.

2. **Random Errors** - errors resulting in the fluctuation of measurements of the same quantity about the average. The measurements are equally probable of being too large or too small. These errors most commonly result from the fineness of scale division of a measuring device.

Some systematic errors can be easy to spot. For example, a device may not "zero" properly, but still have a scale that behaves accurately. An electronic device designed to read current might read 0.01 A, even when the circuit is disconnected and no current can flow. It is obvious in this case that it should read 0 A. Every subsequent measurement of current will be too large by 0.01 A. This is easy to correct. Simple subtract 0.01 A from every

[3]This is the "uncorrected sample standard deviation", sometimes called the "standard deviation of the sample". There is also a "corrected sample standard deviation" which divides by $n-1$ rather than n. For small numbers of measurements, the latter can be the better measure to use, however, for simplicity, we will use the formula given above.

measurement result. Some devices can easily be re-calibrated, to restore their accuracy. In other cases, systematic errors can difficult to identify if there is no available reference to calibrate from. The measurements may have high precision, even though they have low accuracy. In such cases, the systematic error may only become apparent when comparing data to that collected by other groups or other devices or techniques.

Random errors occur because of the inherent limitations on the precision of a measurement device, or due to white noise, or random variation. The effects of this kind of error can be analyzed using error propagation.

Both kinds of errors may result from improper use of a measuring device!

Error propagation for systematic errors. Suppose the value of f depends on the values of x and y. δx and δy are the estimated magnitudes of the systematic errors in x and y, but if we do not know the signs of the errors (whether they make x and y bigger or smaller than they should be) we must assume that in the worst case the errors are "adding up" and we use a more pessimistic formula for finding the uncertainty in f:

$$\delta f = \left|\frac{\partial f}{\partial x}\right| \delta x + \left|\frac{\partial f}{\partial y}\right| \delta y$$

Error propagation for random errors. Again, suppose the value of f depends on the values of x and y. If we can assume the uncertainties in x and y are the result of random errors, then the expected error in f is smaller that the error would be in the worst case. We can use the formula derived for the standard deviations of x and y, using δx and δy in place of the standard deviations:

$$\delta f = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 (\delta x)^2 + \left(\frac{\partial f}{\partial y}\right)^2 (\delta y)^2 + \left(\frac{\partial f}{\partial f}\right)^2 (\delta z)^2}$$

3 Uncertainties from Measurement Devices

We can consider the uncertainties from devices that operate in different ways. Scale measuring devices are analog devices, such as rulers. Digital measuring devices give numbers as outputs and it is not possible to make finer estimates because there is no scale to reference. Two simple rules are:

Uncertainty in a Scale Measuring Device is equal to the smallest increment divided by 2.

$$\delta x = \text{smallest increment } / \ 2$$

Sometimes it may be possible to visually subdivide the smallest increment even further, so that the measurement may be made down to say one-fifth of the smallest increment. An uncertainty of the smallest increment divided by 2 is a safe choice however.

Uncertainty in a Digital Measuring Device is equal to the smallest increment.

$$\delta x = \text{smallest increment}$$

Since it is not possible to compare the measured value to a scale and you do not know the exact design or workings of a digital device, you should assume that the uncertainty of the measurement is the smallest increment.

Some examples,

Meter Stick (scale device)

$$\sigma_x = 1 \text{ mm} = 0.5 \text{ mm} = 0.05 \text{ cm}$$

Digital Balance (digital device)
Readout: 5.7513 kg

$$\sigma_x = 0.0001 \text{ kg}$$

When stating a measurement the uncertainty should be stated explicitly so that there is no question about the uncertainty in the measurement. However, if the is not stated explicitly, an uncertainty is still implied.

For example, if we measure a length of 5.7 cm with a meter stick, this implies that the length can be anywhere in the range $5.65cm \leq L \leq 5.75$ cm. Thus, $L = 5.7$ cm measured with a meter stick implies an uncertainty of 0.05 cm. A common rule of thumb is to take one-half the unit of the last decimal place in a measurement to obtain the uncertainty.

In general, any measurement can be stated in the following preferred form:

$$\text{measurement} = x_{\text{best}} \pm \delta x$$

where x_{best} is the best estimate of measurement and δx is the uncertainty (error) in measurement.

4 Stating Uncertainties

Rule For Stating Uncertainties - Experimental uncertainties should be stated to 1-significant figure.

For example,

$$\begin{aligned} v &= 31.25 \pm 0.034953 \text{ m/s} \\ v &= 31.25 \pm 0.03 \text{ m/s (correct)} \end{aligned}$$

The uncertainty is just an estimate and thus it cannot be more precise (more significant figures) than the best estimate of the measured value.

Rule For Stating Answers The last significant figure in any answer should be in the same place as the uncertainty. For example,

$$\begin{aligned} a &= 1261.29 \pm 200 \text{ cm/s}^2 \\ a &= 1300 \pm 200 \text{ cm/s}^2 \text{ (correct)} \end{aligned}$$

Since the uncertainly is stated to the hundreds place, we also state the answer to the hundreds place. Note that the uncertainty determines the number of significant figures in the answer.

5 Significant Figures

Significant figures are digits that are known in a value. For example, 3.04 m has three significant figures, while 5.870×10^6 m has four significant figures.

Use scientific notation to clearly indicate how many significant figures a number has.

When measurements are added or subtracted, the number of decimal places in the final answer should equal the smallest number of **decimal places** of any term. Consider adding the following masses,

$$\begin{aligned} M &= 256.5895 \text{ g} + 8.1 \text{ g} \\ M &= 264.6895 \text{ g} \\ M &= 264.7 \text{ g (answer)} \end{aligned}$$

When measurements are multiplied or divided, the number of **significant figures** in the final answer should be the same as the term with the lowest number of significant figures.

$$\begin{aligned} L_1 &= 2.2 \text{ cm} \\ L_2 &= 38.2935 \text{ cm} \\ A &= L_1 L_2 \\ &= 84.126900000 \text{ cm}^2 \\ A &= 84 \text{ cm}^2 \text{ (answer)} \end{aligned}$$