Calculating Moments of Inertia

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1 Definitions

The moment of inertia, I of an object for a particular axis is the constant that links the applied torque τ about that axis to the angular acceleration α about that axis. The equation specifying the proportionality is a rotational version of Newton's second law:

$$\boldsymbol{\tau} = I\boldsymbol{\alpha} \tag{1}$$

The moment of inertia is defined as

$$I = \sum_i m_i r_i^2 \tag{2}$$

for a collection of point-like masses m_i each at a distance r_i from the specified axis. It is also defined as

$$I = \int r^2 \,\mathrm{dm} \tag{3}$$

for a continuous distribution of mass.

2 An Example: Moment of Inertia of a Right Circular Cone

For a right circular cone of uniform density we can calculate the moment of inertia by taking an integral over the volume of the cone and appropriately weighting each infinitesimal unit of mass by its distance from the axis squared.

This can be done in several ways.

First let us put some parameters on the problem. Let the radius of the cone's circular base be R and the height of the cone be H. Let the mass of the cone be M and its density be $\rho = \frac{M}{V}$, where $V = \frac{1}{3}\pi R^2 H$ be the volume of the cone.

Using the fact that the density is constant throughout the solid, we will write the definition of the moment of inertia as:

$$\begin{aligned} I &= \int r^2 \,\mathrm{dm} \\ &= \rho \int r^2 \,\mathrm{dV} \end{aligned}$$

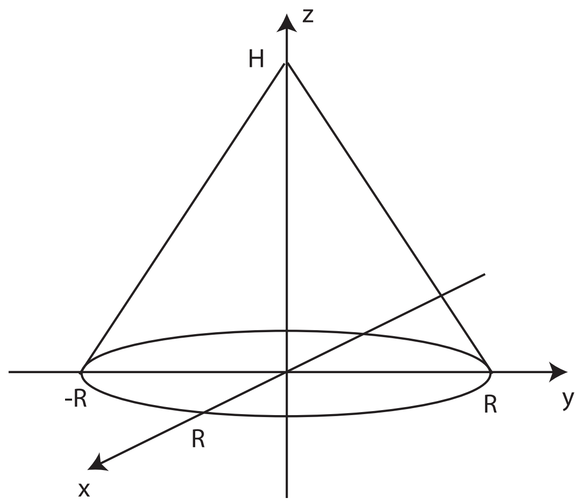

Figure 1: A right circular cone.

2.1 Using symmetry

We can observe that the cone will be made up of cylindrical shells of infinitesimal thickness, each at an equal distance from the axis of rotation. For this calculation, we position the cone in on the axes as show in figures 1 and 2. Because the cone slopes downward, the cylindrical shells each have a different height, and we must take that into account. Each shell will have the same infinitesimal thickness dr.

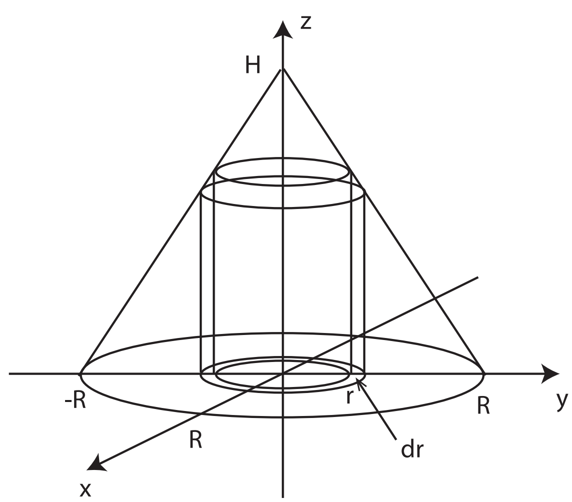

Figure 2: Summing cylindrical shells in a right circular cone.

The height of each cylinder will vary with the radius. Since a cross-section of the cone through the center gives an isosceles triangle, the height of the triangle at a given distance from the line of symmetry varies linearly with the distance (the radius r). This is the height

$z(r)$ of each cylindrical shell at radius r. The height must be 0 at $r = R$ and H at $r = 0$. These constraints give:

$$z(r) = H\left(1 - \frac{r}{R}\right)$$

Then we must find the contribution to the volume of each cylindrical shell. Each shell is an infinitesimally thin sheet of area $A = 2\pi r z(r)$. This is simply the surface area of a cylinder, not including the end caps. The thin sheet's thickness is dr, so we can write that the volume contribution from this thin slice is $\mathrm{dV} = 2\pi r z(r)\,\mathrm{dr}$.

The integral becomes:

$$\begin{aligned} I &= \rho \int r^2 \,\mathrm{dV} \\ &= 2\pi\rho \int_0^R r^3 z(r)\,\mathrm{dr} \end{aligned}$$

Replacing $h(r)$ with its linear function in terms of r and ρ with the ratio of mass to volume:

$$\begin{aligned} I &= \frac{2\pi M}{(1/3)\pi R^2 H} \int_0^R r^3 H\left(1 - \frac{r}{R}\right) \mathrm{dr} \\ &= \frac{6M}{R^2} \int_0^R \left(r^3 - \frac{r^4}{R}\right) \mathrm{dr} \\ &= \frac{6M}{R^2} \left[\frac{r^4}{4} - \frac{r^5}{5R}\right]_0^R \\ &= \frac{6M}{R^2} R^4 \left[\frac{1}{4} - \frac{1}{5}\right] \\ &= \frac{6}{20} M R^2 \\ I &= \frac{3}{10} M R^2 \end{aligned}$$

We could also position the cone as in figure 3. In that case nothing changes in this evaluation, as we would still have $z(r) = H\left(1 - \frac{r}{R}\right)$.

2.2 Triple integral

Another fine way to evaluate this is using the triple integral for volume: just be sure to get the integration limits right!

The radial symmetry in this problem makes cylindrical coordinates the best choice for the triple integral. In cylindrical coordinates, $\mathrm{dV} = r\,\mathrm{dr}\,\mathrm{d}\phi\,\mathrm{dz}$.

Now the moment of inertia integral becomes:

$$\begin{aligned} I &= \rho \int r^2 \,\mathrm{dV} \\ &= \rho \iiint r^3 \,\mathrm{dr}\,\mathrm{d}\phi\,\mathrm{dz} \end{aligned}$$

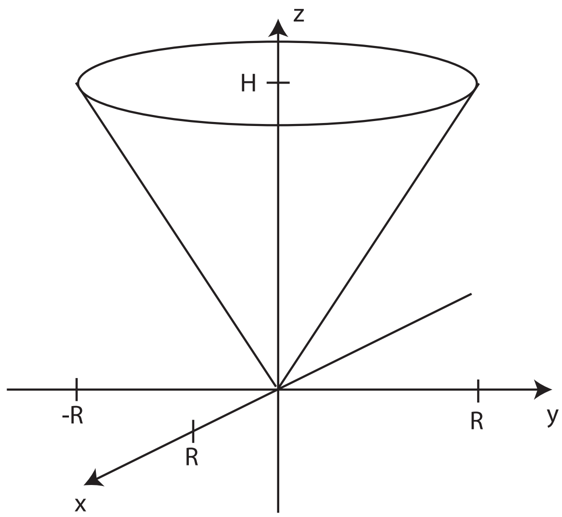

Figure 3: The same cone, in a different orientation on the coordinate axes.

The limits of each integral must be determined. For the ϕ integral, the cone has full radial symmetry, so we must sum up the contributions for all masses distributed around the full circle. This gives us a range of $[0, 2\pi)$, and is independent of the other variables. For the other two integrals, we can do them in either order, but it will affect the limits of each.

First, let us evaluation doing the z integral first, then the r integral. This should look very similar to what happens in section 2.1.

$$I = \frac{M}{(1/3)\pi R^2 H} \int_0^R \int_0^{H\left(1-\frac{r}{R}\right)} \int_0^{2\pi} \mathrm{d}\phi \, r^3 \, \mathrm{dz} \, \mathrm{dr}$$

Notice that the upper bound on the z integral is our function defining the boundary of the cone.

$$\begin{aligned} I &= \frac{3M}{\pi R^2 H} \int_0^R \int_0^{H\left(1-\frac{r}{R}\right)} [\phi]_0^{2\pi} \, r^3 \, \mathrm{dz} \, \mathrm{dr} \\ &= \frac{6\pi M}{\pi R^2 H} \int_0^R \left[\; z \; \right]_0^{H\left(1-\frac{r}{R}\right)} r^3 \, \mathrm{dr} \\ &= \frac{6M}{R^2 H} \int_0^R H r^3 \left(1 - \frac{r}{R}\right) \mathrm{dr} \\ &= \frac{6M}{R^2} \left[\frac{r^4}{4} - \frac{r^5}{5R} \right]_0^R \\ &= \frac{6M}{R^2} R^4 \left[\frac{1}{4} - \frac{1}{5} \right] \\ I &= \frac{3}{10} M R^2 \end{aligned}$$

Note: Be careful to integrate over the correct bounds. The correct bounds can be different depending on how you have arranged your solid object in coordinate space.

As an example of this, consider the cone as drawn in figure 3. If we specify the cone in this way, the bounds on the z-integral will change:

$$
\begin{aligned}
I &= \frac{3M}{\pi R^2 H}\int_0^R \int_{\frac{rH}{R}}^{H} [\phi]_0^{2\pi}\, r^3 \,\mathrm{dz}\,\mathrm{dr} \\
&= \frac{6\pi M}{\pi R^2 H}\int_0^R \Big[\ z\ \Big]_{\frac{rH}{R}}^{H} r^3\,\mathrm{dr} \\
&= \frac{6M}{R^2 H}\int_0^R Hr^3\left(1-\frac{r}{R}\right)\mathrm{dr} \\
I &= \frac{3}{10}MR^2
\end{aligned}
$$

Notice that this leads to the same result. However, if we had used z-integral bounds $[0, rH/R]$, we would have been evaluating the wrong shape and would have gotten the wrong answer.

Now, returning to the arrangement of figures 1 and 2. We could also evaluate this doing the integral over r first, but then we have to make sure to give the correct limits on the integral. We will need to invert our function relating r to z: $r(z) = R(1 - z/H)$

$$
\begin{aligned}
I &= \frac{M}{(1/3)\pi R^2 H}\int_0^H \int_0^{R(1-z/H)} \int_0^{2\pi} \mathrm{d}\phi\, r^3\,\mathrm{dr}\,\mathrm{dz} \\
&= 2\pi\frac{3M}{\pi R^2 H}\int_0^H \int_0^{R(1-z/H)} r^3\,\mathrm{dr}\,\mathrm{dz} \\
&= \frac{6M}{R^2 H}\int_0^H \left[\frac{r^4}{4}\right]_0^{R(1-z/H)} \mathrm{dz} \\
&= \frac{6M}{R^2 H}\frac{R^4}{4}\int_0^H \left(1-\frac{z}{H}\right)^4 \mathrm{dz} \\
&= \frac{3MR^4}{2H}\int_0^H \left(1-\frac{z}{H}\right)^4 \mathrm{dz} \\
&= \frac{3MR^4}{2H}\left[\frac{-H}{5}\left(1-\frac{z}{H}\right)^5\right]_0^H \\
&= \frac{3MR^4}{2}\left[0+\frac{1}{5}\right] \\
I &= \frac{3}{10}MR^2
\end{aligned}
$$

2.3 Adding circular contributions

This way is a bit less intuitive perhaps, but can also work.

Here we observe that a cone can be seen as a stack of infinitesimally thin circular disks. Let's look at how that would go if we simply wanted to evaluate the *volume* of the cone (not the moment of inertia!). Each circular volume element will have a cross-sectional area πr^2

and a thickness dz The integral becomes:

$$\begin{aligned} V &= \int \mathrm{dV} \\ &= \pi \int_0^H r^2 \, \mathrm{dz} \end{aligned}$$

r is a function of z, so replacing $r(z) = R\left(1 - \frac{z}{H}\right)$:

$$\begin{aligned} V &= \pi \int_0^H R^2 \left(1 - \frac{z}{H}\right)^2 \mathrm{dz} \\ &= \pi R^2 \left[\frac{-H}{3}\left(1 - \frac{z}{H}\right)^3\right]_0^H \\ &= \pi R^2 \left[0 + \frac{H}{3}\right] \\ &= \frac{1}{3}\pi R^2 H \end{aligned}$$

That works fine.

For the moment of inertia, we want to weight all the mass at a radius r with a factor of r^2 when we do the integral to sum over it. If we leap to the summing up circular disks of area πr^2, when we multiply in a factor of r^2, we will be weighting each circle of mass according to its radius, rather than each infinitesimal mass according to its distance from the axis. Most of the material in the circular disk is in the interior of the disk, not along the circumference! Naïvely doing this integral will cause us to *overestimate* the moment of inertia.

What we must do instead is add up all the moments of inertia of these thin disks to get the total moment of inertia of the cone (a stack of disks).

A disk of radius r and mass m has moment of inertia $I_d = \frac{1}{2}mr^2$ (d for "disk"). Then we allow that each disk is infinitesimally thin, and contributes a mass dm: $\mathrm{dI}_d = \frac{1}{2}r^2 \, \mathrm{dm}$.

So, in a moment of inertia calculation, we add up the contributions from thin circular disks to the total moment of inertia. In the third line we will again use that $\rho = \frac{M}{V} = \frac{M}{(1/3)\pi R^2 H}$.

$$\begin{aligned}
I &= \int \mathrm{dI}_d \\
&= \int \frac{1}{2} r^2 \,\mathrm{dm} \\
&= \int \frac{1}{2} r^2 (\pi r^2 \rho) \,\mathrm{dz} \\
&= \left(\frac{1}{2}\right) \frac{M}{(1/3)\pi R^2 H} \int_0^H \pi r^4 \,\mathrm{dz} \\
&= \frac{3M}{2R^2 H} \int_0^H \left(1 - \frac{z}{H}\right)^4 \mathrm{dz} \\
&= \frac{3M}{2R^2 H} \left[\frac{-H}{5}\left(1 - \frac{z}{H}\right)^5\right]_0^H \\
&= \frac{3MR^4}{10} [0 + 1] \\
I &= \frac{3}{10} M R^2
\end{aligned}$$

Notice that this worked because we already assumed that we knew the moment of inertia of a disk, even before we began the calculation. A double integral would be required to prove that statement $I_d = \frac{1}{2} m r^2$, we are simply omitting that step.

In general it is often more clear to do the triple integral.