Dynamics
Applying Newton's Laws
The Drag Equation

Lana Sheridan

De Anza College

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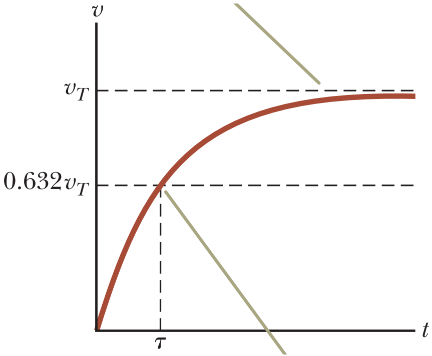
Last time

- introduced resistive forces
- model 1: Stokes drag

Recap: Model 1 - Stokes Drag

Exercise: Check that the solution $v(t) = \frac{mg}{b}(1 - e^{-bt/m})$ satisfies the equation

$$m\,\frac{\mathrm{dv}}{\mathrm{dt}} = mg - bv$$



Time constant, $\tau = \frac{m}{b}$. The solution can also be written $v(t) = v_T(1 - e^{-t/\tau})$.

Overview

- model 2: the Drag Equation
- finish resistive forces

Resistive Forces

There are two main models for how this happens. Either the resistive force $\vec{\mathbf{R}}$

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Model 2: The Drag Equation

For this model $R \sim v^2$.

Appropriate for high Reynolds Number fluids and/or objects moving at high speed.

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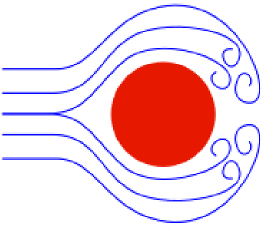
There is (or can easily be) some turbulence in these cases.



Model 2: The Drag Equation

Altogether, the conditions for use are:

- fast moving objects
- low viscousity fluids
- turbulent flow in the fluid

Model 2: The Drag Equation

The magnitude of the resistive force is:

$$R = \frac{1}{2} D \rho A v^2$$

This is called the *drag equation*.

- D is the *drag coefficent* (depends on object's shape)
- ρ is the density of air
- A is the cross-sectional area of the object perpendicular to its motion

Model 2: The Drag Equation

It is straightforward to find an expression for v_T in this case also. Equilibrium $\Rightarrow$ $F_{\text{net}} = 0$.

$$\begin{aligned} F_{\text{net}} = \frac{1}{2} D\rho A v_T^2 - mg &= 0 \\ \frac{1}{2} D\rho A v_T^2 &= mg \end{aligned}$$

$$v_T = \sqrt{\frac{2mg}{D\rho A}}$$

Model 2: The Drag Equation

(We are skipping this derivation in lecture.)

Finding the net force and using Newton's second law ($\downarrow +y$):

$$\begin{aligned} F_{\text{net},y} &= ma_y \\ &\swarrow\!\!\searrow \\ ma &= mg - \frac{1}{2} D\rho A v^2 \\ \frac{1}{g}\frac{\mathrm{dv}}{\mathrm{dt}} &= 1 - \frac{D\rho A}{2mg} v^2 \end{aligned}$$

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Let $k = \sqrt{\frac{D\rho A}{2mg}} = \frac{1}{v_T}$.

$$\frac{1}{g}\frac{\mathrm{dv}}{\mathrm{dt}} = 1 - k^2 v^2$$

Model 2: The Drag Equation

(We are skipping this derivation in lecture.)

We can separate variables[1] and integrate with respect to t:

$$\begin{aligned}\frac{1}{g}\,\frac{\mathrm{dv}}{\mathrm{dt}} &= 1 - k^2 v^2 \\ \frac{1}{g}\int_0^v \frac{1}{1-k^2v^2}\,\frac{\mathrm{dv}}{\mathrm{dt}}\,\mathrm{dt} &= \int_0^t 1\,\mathrm{dt}'\end{aligned}$$

[1] The integrating factor trick does not work cleanly here.

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\frac{1}{g}\int_0^v \frac{1}{1-k^2v^2}\,\mathrm{dv} &= t
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We need to find $\int \frac{1}{1-k^2v^2}\mathrm{dv}$.
Use partial fractions:

$$\frac{1}{1-k^2v^2} = \frac{1}{(1+kv)(1-kv)} = \frac{1}{2}\left(\frac{1}{(1+kv)} + \frac{1}{(1-kv)}\right)$$

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Model 2: The Drag Equation

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After integrating and some algebra:

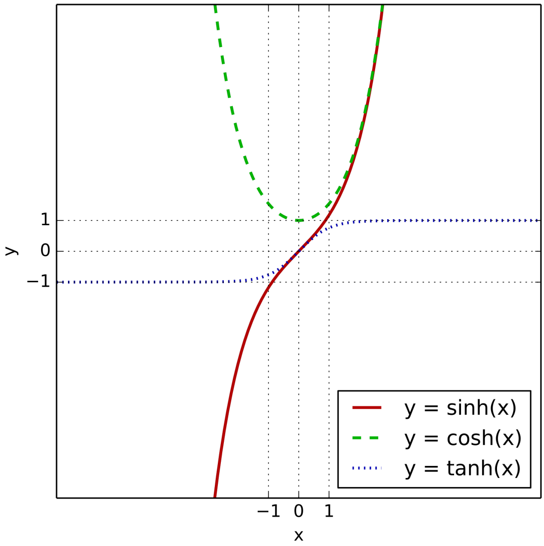
$$v(t) \quad = \quad \frac{1}{k}\left(\frac{1-e^{-2kgt}}{1+e^{-2kgt}}\right)$$

We can write this a little more conveniently as a hyperbolic function.

$$v(t) = v_T \tanh\left(\frac{g}{v_T} t\right)$$

or $v(t) = \sqrt{\frac{2mg}{D\rho A}} \tanh\left(\sqrt{\frac{D\rho A g}{2m}}\, t\right)$.

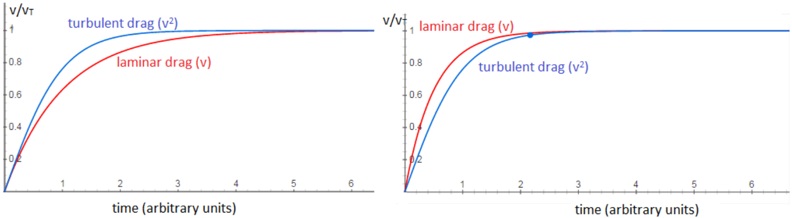
Hyperbolic functions



[1]Figure by Wikipedia user Fylwind.

Comparing the two models

The two types of resistive forces predict different curves, but depending on the constants, in each model the curve might approach the terminal velocity faster or slower.

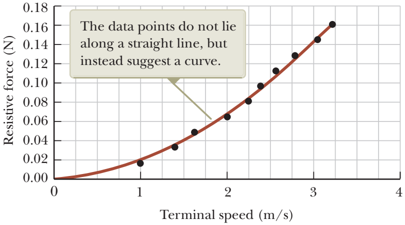


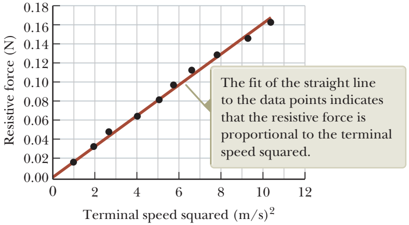
[$(1 - e^{-x})$ vs. $\tanh x$]

[1] Figure from Wikipedia.

Determining the regime: which model?

Experimental data:





The resistive force $R = mg$ at the terminal velocity. Plot R against v_T.

[1] Figures from Serway and Jewett.

One more point about resistive forces

What if the object is not dropped from rest?

If an object starts out with a large velocity, then begins to experience an unbalanced resistive force it will have an acceleration opposite to its direction of motion.

Consider what happens during a skydive.

Drag Equation Example

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27. The mass of a sports car is 1 200 kg. The shape of the body is such that the aerodynamic drag coefficient is 0.250 and the frontal area is 2.20 m^2. Ignoring all other sources of friction, calculate the initial acceleration the car has if it has been traveling at 100 km/h and is now shifted into neutral and allowed to coast.

(You can use $\rho_{air} = 1.20$ kg m^{-3}.)

Drag Equation Example

Page 171, #27

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Let the car's direction of travel be $+x$.

$$F_{net,y} = m\cancelto{0}{a_y}$$

$$\begin{aligned} F_{net,x} &= ma_x \\ -\frac{1}{2}D\rho A v^2 &= ma \\ a &= -\frac{1}{2m}D\rho A v^2 \end{aligned}$$

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Page 171, #27

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Summary

- Drag equation resistive force model

Test Monday.

(Uncollected) Homework

- study!

Serway & Jewett,

- Ch 6, onward from page 171. Probs: 33, 66