Linear Momentum
Isolated Systems: Conservation of Momentum

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Last time

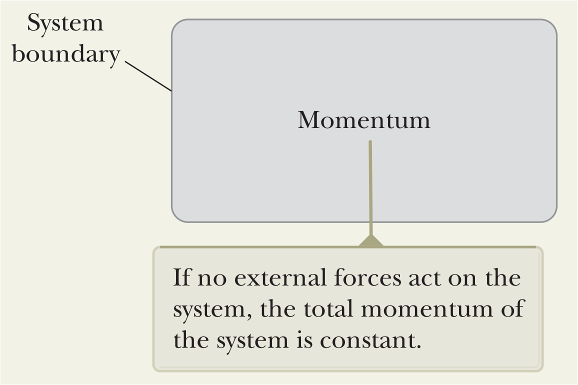
- introduced momentum
- Newton's Second Law: more general form
- relation to force
- momentum vs kinetic energy

Overview

- conservation of momentum
- relation to Newton's third law
- the rocket equation
- applying the rocket equation
- conservation of momentum in isolated systems

Conservation of Linear Momentum

For an *isolated* system, *ie.* a system with **no external forces**, total linear momentum is *conserved*.



[1]Figures from Serway & Jewett.

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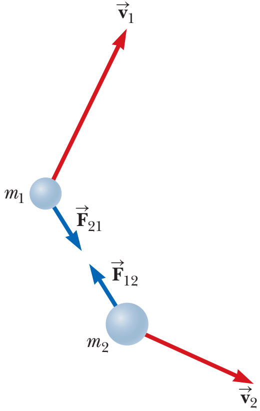
(Note: before when speaking of energy "isolated" meant "not exchanging energy" now, for momentum, it means, no external forces act on the system.)

Newton's Third Law and Conservation of Momentum



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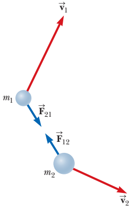
$$\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$$

Newton's Third Law and Conservation of Momentum



Newton's third law for two interacting particles:

$$\begin{aligned} \vec{\mathbf{F}}_{21} &= -\vec{\mathbf{F}}_{12} \\ \frac{d\vec{\mathbf{p}}_1}{dt} &= -\frac{d\vec{\mathbf{p}}_2}{dt} \end{aligned}$$

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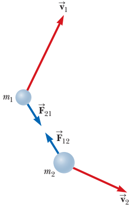

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$$\begin{aligned} \vec{\mathbf{F}}_{21} &= -\vec{\mathbf{F}}_{12} \\ \frac{\mathrm{d}\vec{\mathbf{p}}_1}{\mathrm{d}t} &= -\frac{\mathrm{d}\vec{\mathbf{p}}_2}{\mathrm{d}t} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2\right) &= 0 \end{aligned}$$

Implies:
$\vec{\mathbf{p}}_{\text{net}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2$ does not change with time. Or, $\Delta\vec{\mathbf{p}}_{\text{net}} = 0$.

Newton's Third Law and Conservation of Momentum

Newton's third law $\Leftrightarrow$ conservation of momentum

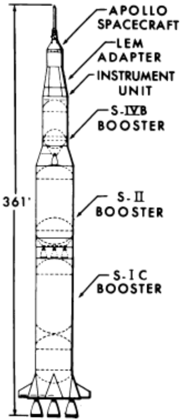
No external forces (only internal action-reaction pairs):

$$\Delta \vec{\mathbf{p}}_{\text{net}} = 0$$

The Rocket Equation

A case where the mass is changing.

A famous example: what happens to a rocket as it burns fuel. ("The rocket equation")



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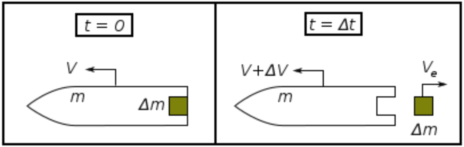
The Rocket Equation

Main idea: conserve momentum between the rocket and the ejected propellant.



Suppose that the rocket burns fuel at a steady rate with respect to time, and let the initial mass of the rocket and propellent be m_i (at time t_i) and the final mass be m_f (at t_f).

Newton's third law:

$$\vec{\mathbf{F}}_{\text{rocket}} = -\vec{\mathbf{F}}_{\text{exhaust}}$$

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[0]Figure by Wikipedia user Skorkmaz.

The Rocket Equation

Consider this interaction **in the frame of the rocket** and suppose the force F_r on the rocket is constant with time.

Newton's Second Law for the rocket:

$$\vec{\mathbf{F}}_{e\to r} = \frac{\mathrm{d}\,\vec{\mathbf{p}}}{\mathrm{dt}} = m\,\frac{\mathrm{d}\,\vec{\mathbf{v}}}{\mathrm{dt}} + \vec{\mathbf{v}}\,\frac{\mathrm{dm}}{\mathrm{dt}}$$

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In the rocket frame, the rocket is not moving ($\vec{\mathbf{v}} = 0$).

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Constant force $\Rightarrow$ constant exhaust velocity in the frame of the rocket. ($\frac{\mathrm{dv_e}}{\mathrm{dt}} = 0$)

The Rocket Equation

So, if $\vec{\mathbf{v}}$ is the velocity of the rocket

$$\begin{aligned} -\vec{\mathbf{F}}_{e\to r} &= \vec{\mathbf{F}}_{r\to e} \\ -\vec{\mathbf{v}}_{\mathbf{e}}\,\frac{\mathrm{dm_e}}{\mathrm{dt}} &= m\,\frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{dt}} \end{aligned}$$

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but $\frac{\mathrm{dm_e}}{\mathrm{dt}} = -\frac{\mathrm{dm}}{\mathrm{dt}}$ since the rate mass in ejected is equal to the rate of *loss* of rocket mass:

$$\vec{\mathbf{v}}_{\mathbf{e}}\frac{\mathrm{dm}}{\mathrm{dt}} = m\frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{dt}}$$

($\frac{\mathrm{dm}}{\mathrm{dt}}$ is $-$ve, $\vec{\mathbf{v}}_{\mathbf{e}}$ and $\frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{dt}}$ point in opposite dirs.)

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Consider the x-axis to drop the vector notation, noticing $\vec{\mathbf{v}}_{\mathbf{e}}$ points in the opposite direction of $\vec{\mathbf{v}}$ and $\frac{\mathrm{d}\vec{\mathbf{v}}}{\mathrm{dt}}$. Therefore,

$$(-v_e \,\hat{\mathbf{i}}) \frac{\mathrm{dm}}{\mathrm{dt}} = m \left(\frac{\mathrm{dv}}{\mathrm{dt}} \,\hat{\mathbf{i}} \right)$$

The Rocket Equation

Rocket's change in velocity, Δv?

$$\frac{\mathrm{dv}}{\mathrm{dt}} = -\frac{v_e}{m}\frac{\mathrm{dm}}{\mathrm{dt}}$$

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$$\begin{aligned} \frac{\mathrm{dv}}{\mathrm{dt}} &= -\frac{v_e}{m}\,\frac{\mathrm{dm}}{\mathrm{dt}} \\ \Delta v &= -\int_{t_i}^{t_f} \frac{v_e}{m(t)}\,\frac{\mathrm{dm}}{\mathrm{dt}}\,\mathrm{dt} \end{aligned}$$

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\end{aligned}$$

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&= -\int_{m_i}^{m_f} \frac{v_e}{m}\,\mathrm{dm} \\
&= -v_e\Big[\ln(m)\Big]_{m_i}^{m_f} \\
\Delta v &= v_e \ln\left(\frac{m_i}{m_f}\right)
\end{aligned}$$

(the "Tsiolkovsky rocket equation")

The Rocket Equation

For a rocket burning fuel at a constant rate:

$$v_f = v_i + v_e \ln\left(\frac{m_i}{m_f}\right)$$

where m_i is the initial mass of the rocket and m_f is the final (smaller) mass.

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The *thrust* on an object is the forward force on the object generated by engines / a propulsion system.

For a contant velocity of the exhaust, v_e :

$$\text{Thrust} = \left| v_e \, \frac{\mathrm{dm}}{\mathrm{dt}} \right|$$

Example 9.17

A rocket moving in space, far from all other objects, has a speed of 3.0×10^3 m/s relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket.[1]

(a) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to half its mass before ignition?

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

[3]Serway & Jewett, page 279.

Example 9.17

(a) Speed with half of mass gone? Let initial mass be m_i

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v_f &= v_i + v_e \ln\left(\frac{m_i}{m_f}\right) \\
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v_f &= \underline{6.5 \times 10^3 \text{ m/s}}
\end{aligned}$$

(b) Thrust if $\left|\frac{\mathrm{dm}}{\mathrm{dt}}\right| = 50$ kg/s?

$$F_{\text{Thrust}} = \left| v_e \frac{\mathrm{dm}}{\mathrm{dt}} \right|$$

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$$\begin{aligned}
F_{\text{Thrust}} &= \left|v_e\frac{\text{dm}}{\text{dt}}\right| \\
F_{\text{Thrust}} &= (5.0\times10^3)(50) \\
F_{\text{Thrust}} &= \underline{2.5\times10^5\ \text{N}}
\end{aligned}$$

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When discussing energy, we ignored the kinetic energy of the Earth when considering the energy of a system consisting of the Earth and a dropped ball. ($\Delta K_{\text{ball}} + \Delta U_g = 0.$) Verify this is a reasonable thing to do.

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Momentum is conserved when we drop a ball, if we include the Earth in our system: $\Delta \vec{\mathbf{p}}_b + \Delta \vec{\mathbf{p}}_E = 0$.

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$$\begin{aligned}
\Delta \vec{\mathbf{p}}_b &= -\Delta \vec{\mathbf{p}}_E \\
m_b(\vec{\mathbf{v}}_{b,f} - 0) &= -m_E(\vec{\mathbf{v}}_{E,f} - 0) \\
m_b v_{b,f} &= m_E v_{E,f} \\
\frac{v_{E,f}}{v_{b,f}} &= \frac{m_b}{m_E}
\end{aligned}$$

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$$\begin{aligned}
\frac{\Delta K_{\text{Earth}}}{\Delta K_{\text{ball}}} &= \frac{\frac{1}{2} m_E v_{E,f}^2}{\frac{1}{2} m_b v_{b,f}^2} \\
&= \frac{m_E}{m_b} \left(\frac{v_{E,f}}{v_{b,f}} \right)^2 \\
&= \frac{m_E}{m_b} \left(\frac{m_b}{m_E} \right)^2 \\
&= \frac{m_b}{m_E}
\end{aligned}$$

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&= \frac{m_E}{m_b} \left(\frac{v_{E,f}}{v_{b,f}} \right)^2 \\
&= \frac{m_E}{m_b} \left(\frac{m_b}{m_E} \right)^2 \\
&= \frac{m_b}{m_E} \\
&= \frac{1 \text{ kg}}{10^{25} \text{ kg}} << 1
\end{aligned}$$

Summary

- conservation of momentum and Newton's third law
- the rocket equation
- using the rocket equation

(Uncollected) Homework Serway & Jewett,

- Read along in Chapter 9.
- Ch 9, onward from page 283. Probs: 1, 3, 5, 7, 61, 63