Extended or Composite Systems
Center of Mass

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Last time

- center of mass
- finding the center of mass
- center of mass for continuous mass distributions

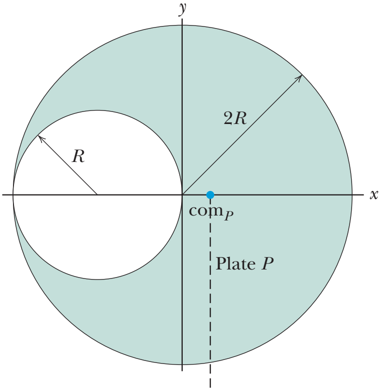
Overview

- center of mass examples
- center of mass integral
- collective motion of systems of particles

Center of Mass of Continuous Objects

Suppose this 2-D object, plate P, has uniform mass-per-unit-area.

How can we find its center of mass without integrating?



Center of Mass of Continuous Objects

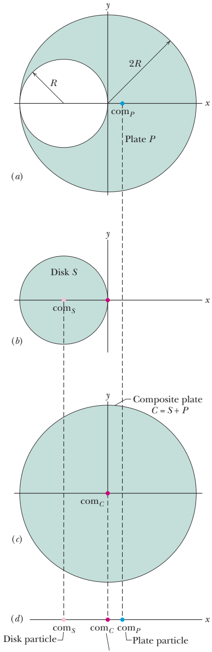
Let the mass-per-unit-area be σ.

Mass of little disk S, $M_S = \pi R^2 \sigma$.

Mass of composite C, $M_C = \pi (2R)^2 \sigma$.

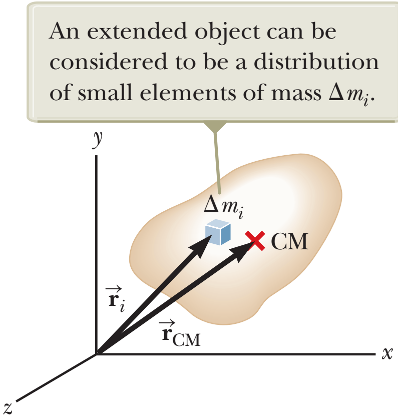
Mass of plate P,
$M_P = M_C - M_S = 3\pi R^2 \sigma$.

$$
\begin{aligned}
M_C\, x_{CM,C} &= M_P\, x_{CM,P} + M_S\, x_{CM,S} \\
x_{CM,P} &= \frac{M_C\, x_{CM,C} - M_S\, x_{CM,S}}{M_P} \\
x_{CM,P} &= \frac{\pi (2R)^2 \sigma\,(0) - \pi R^2 \sigma\,(-R)}{3\pi R^2 \sigma} \\
x_{CM,P} &= \frac{R}{3}
\end{aligned}
$$



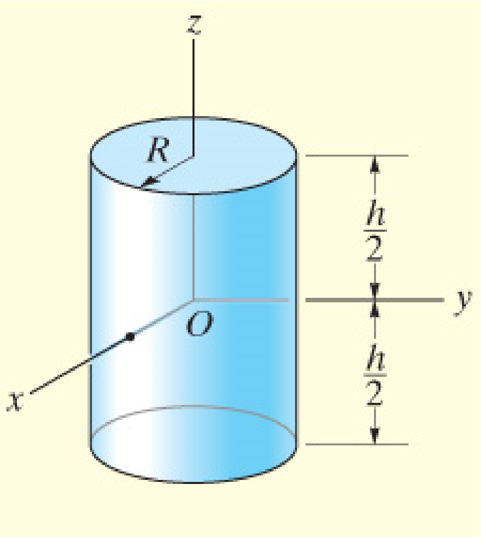
Continuous mass distribution

$$\vec{\mathbf{r}}_{CM} = \frac{1}{M}\int \vec{\mathbf{r}}\, dm$$

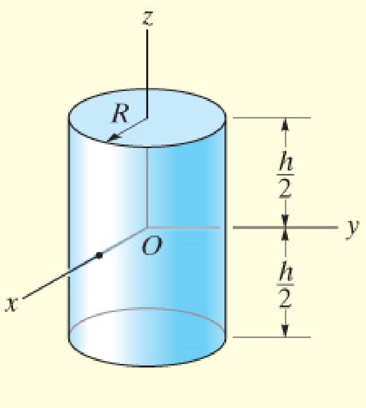


Continuous mass distribution: Another Example

Center of mass of a cylinder of uniform density, ρ.



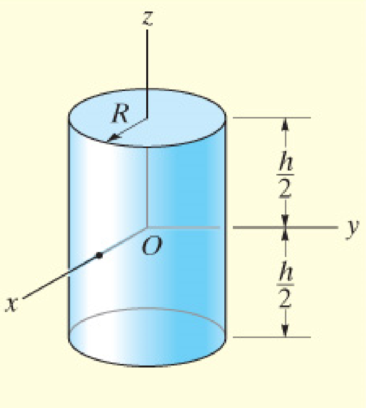
Continuous mass distribution: Another Example



Let's choose our axes so that z points along the length of the cylinder, and the origin is right in the center of the cylinder.

Probably, you can easily guess where the CM will be.

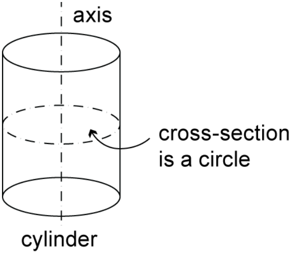
Continuous mass distribution: Another Example



Let's choose our axes so that z points along the length of the cylinder, and the origin is right in the center of the cylinder.

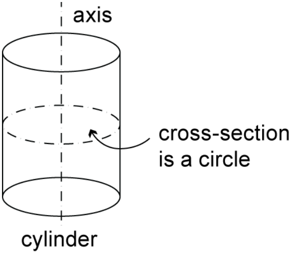
Probably, you can easily guess where the CM will be. At the origin!

Continuous mass distribution: Another Example



How could we prove it?

Continuous mass distribution: Another Example



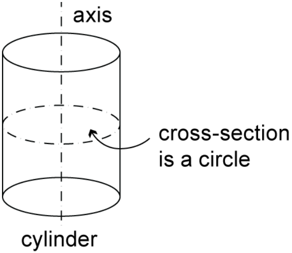
How could we prove it?

Along the z-direction this cylinder is very similar to the thin rod we just considered.

Observe that $\mathrm{dm} = (\pi R^2)\rho\,\mathrm{dz}$:

$$
\begin{aligned}
z_{\mathrm{CM}} &= \frac{1}{M}\int z(\pi R^2 \rho\,\mathrm{dz}) \\
&= \frac{(\pi R^2)\rho}{\pi R^2 h\rho}\int_{-h/2}^{h/2} z\,\mathrm{dz}
\end{aligned}
$$

Continuous mass distribution: Another Example



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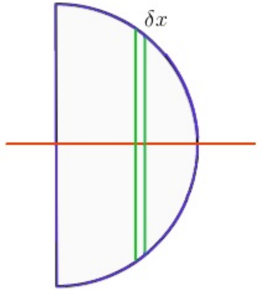
Continuous mass distribution: Another Example

Along the x-direction this cylinder we are looking down on a circle. Let $x = R\cos\theta$, $y = R\sin\theta$

Observe that $\mathrm{dm} = (2yh)\rho\,\mathrm{dx}$:

(Looking at just the positive x part of the cylinder, viewed top down.)



[1]Or, leave the integral in terms of x and use the substitution $u = x^2$.

Continuous mass distribution: Another Example

Along the x-direction this cylinder we are looking down on a circle.
Let $x = R\cos\theta$, $y = R\sin\theta$

Observe that $\mathrm{dm} = (2yh)\rho\,\mathrm{dx}$:

$$
\begin{aligned}
x_{\mathrm{CM}} &= \frac{1}{M}\int x(2yh)\rho\,\mathrm{dx} \\
&= \frac{h\rho}{\pi R^2 h}\int_{\pi}^{0} R^2(2\sin\theta\cos\theta)(-R\sin\theta\,\mathrm{d}\theta) \\
&= \frac{\rho}{\pi R^2}\left[-\frac{2}{3}R^3\sin^3\theta\right]_{\pi}^{0} \\
&= 0
\end{aligned}
$$

---

[1] Or, leave the integral in terms of x and use the substitution $u = x^2$.

Continuous mass distribution: Another Example

By symmetry the evaluation for y_{CM} must go exactly the same way as for x_{CM}, therefore,

$$y_{\text{CM}} = 0\,.$$

This is what we expected all along, but this same technique can be used in cases where we cannot guess the answer.

Systems of Particles

Previously in the course, we had the CM point standing in for an extended object to model the object as point-like.

Now we justify why that works.

(Here we are just confirming things we were already assuming.)

Motion of the Center of Mass

$$\vec{\mathbf{r}}_{\mathrm{CM}} = \frac{1}{M} \sum_i m_i \vec{\mathbf{r}}_i$$

where $\vec{\mathbf{r}}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\vec{\mathbf{k}}$ is the displacement of particle i from the origin.
Differentiating gives:

$$\begin{aligned} \frac{\mathrm{d}\vec{\mathbf{r}}_{\mathrm{CM}}}{\mathrm{dt}} &= \frac{1}{M} \sum_i m_i \frac{\mathrm{d}\vec{\mathbf{r}}_{\mathrm{i}}}{\mathrm{dt}} \\ \vec{\mathbf{v}}_{\mathrm{CM}} &= \frac{1}{M} \sum_i m_i \vec{\mathbf{v}}_i \end{aligned}$$

And differentiating one more time:

$$\vec{\mathbf{a}}_{\mathrm{CM}} = \frac{1}{M} \sum_i m_i \vec{\mathbf{a}}_i$$

Newton's 2nd Law, revisited

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Newton's 2nd Law, revisited

$$\vec{\mathbf{a}}_{\text{CM}} = \frac{1}{M} \sum_i m_i \vec{\mathbf{a}}_i$$

Newton's 2nd law for a particle i tells us $\sum_j \vec{\mathbf{F}}_{j,i} = m_i \vec{\mathbf{a}}_i$, where $\sum_j \vec{\mathbf{F}}_{j,i}$ is the sum of all forces on particle i.

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Then,

$$\vec{\mathbf{a}}_{\text{CM}} = \frac{1}{M} \sum_{i,j} \vec{\mathbf{F}}_{j,i}$$

and $\sum_{i,j} \vec{\mathbf{F}}_{j,i}$ is the sum over all forces on all particles. It's the net force!
(Note that internal forces cancel out and we can ignore them.)

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Therefore,

$$\vec{\mathbf{F}}_{\mathrm{net}} = M \vec{\mathbf{a}}_{\mathrm{CM}}$$

Newton's 2nd Law, revisited

$$\vec{\mathbf{F}}_{\text{net}} = M\vec{\mathbf{a}}_{\text{CM}}$$

Newton's 2nd law holds for the acceleration of the center of mass. This is good because we've already been assuming it when we treated blocks as point masses.

Notice that when we said: $\vec{\mathbf{F}}_{\text{net}} = \sum_{i,j} \vec{\mathbf{F}}_{j,i}$, we did not care which particle / part of the system each external force acted on. It doesn't matter! So, we can treat all external forces as acting at the center of mass of the system.

Center of Mass Motion and Total Momentum

Looking at the velocity equation:

$$\vec{\mathbf{v}}_{\mathrm{CM}} = \frac{1}{M} \sum_i m_i \vec{\mathbf{v}}_i$$

multiply both sides by M:

$$M\, \vec{\mathbf{v}}_{\mathrm{CM}} = \sum_i \vec{\mathbf{p}}_i = \vec{\mathbf{p}}_{\mathrm{net}}$$

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The net momentum of an entire system of many particles can be found by multiplying the total mass and the velocity of the center of mass.

$$M\,\vec{\mathbf{v}}_{\mathrm{CM}} = \vec{\mathbf{p}}_{\mathrm{net}}$$

Implication: **the center of mass** ($\vec{\mathbf{v}}_{\mathrm{CM}} = 0$) **frame is the same as the center of momentum** ($\vec{\mathbf{p}}_{\mathrm{net}} = 0$) **frame**.
Also, for an isolated system, $\vec{\mathbf{v}}_{\mathrm{cm}}$ does not change in a collision!

Impulse on a System of Particles

Lastly, we still have that the impulse is the total change in momentum of the entire system collectively.

$$
\begin{aligned}
\vec{\mathbf{I}} &= \int \vec{\mathbf{F}}_{\text{net,ext}}\,\mathrm{dt} \\
&= \int M\,\frac{\mathrm{d}\,\vec{\mathbf{v}}_{\text{CM}}}{\mathrm{dt}}\,\mathrm{dt} \\
&= M\int_{v_i}^{v_f} \mathrm{d}\,\vec{\mathbf{v}}_{\text{CM}} \\
&= M\,\Delta\vec{\mathbf{v}}_{\text{CM}}
\end{aligned}
$$

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&= M \int_{v_i}^{v_f} \mathrm{d}\vec{\mathbf{v}}_{\text{CM}} \\
&= M\,\Delta\vec{\mathbf{v}}_{\text{CM}} \\
&= \Delta\vec{\mathbf{p}}_{\text{tot}}
\end{aligned}
$$

Summary

- center of mass for continuous mass distributions
- systems of many particles / collective motion

Quiz Friday.

(Uncollected) Homework Serway & Jewett,

- Look at example 9.12 and make sure you understand it.
- Ch 9, onward from page 288. Probs: 47
- Read Chapter 9, if you haven't already.
- the Extra HW Problem (next slide)[1]

[1] Ans: $x = -0.27R$, if x runs thru C, C'.

Extra HW Problem

A uniform circular plate of radius $2R$ has a circular hole of radius R cut out of it. The center C' of the smaller circle is a distance $0.80R$ from the center C of the larger circle, as shown. What is the position of the center of mass of the plate?