Rotational Motion
Rotational Kinematics
Torque

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Last time

- deforming systems
- rotation
- rotational kinematics

Overview

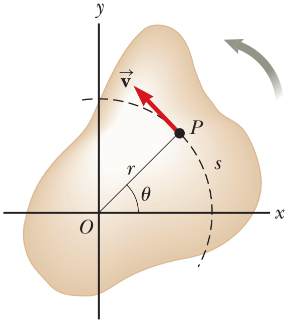
- relating rotational and translational quantities
- torque

Comparison of Linear and Rotational quantities

Linear Quantities	Rotational Quantities
$\vec{\mathbf{x}}$	$\vec{\theta}$
$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{x}}}{dt}$	$\vec{\omega} = \frac{d\vec{\theta}}{dt}$
$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt}$	$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

Relating Rotational Quantities to Translation of Points

Consider a point on the rotating object. How does its speed relate to the angular speed?



We know $s = r\theta$, so since the object's speed is its speed along the path s,

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Relating Rotational Quantities to Translation of Points

Since $\omega = \frac{d\theta}{dt}$, that gives us and expression for (tangential) speed

$$v = r\omega$$

And differentiating both sides with respect to t again:

$$a_t = r\alpha$$

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And differentiating both sides with respect to t again:

$$a_t = r\alpha$$

Notice that the above equation gives the rate of change of speed, which is the tangential acceleration.

Centripetal Acceleration

Remember:

$$a_t = \frac{\mathrm{dv}}{\mathrm{dt}}$$

where v is the speed, not velocity.

So,

$$a_t = r\alpha$$

But of course, in order for a mass at that point, radius r, to continue moving in a circle, there must be a centripetal component of acceleration also.

$$a_c = \frac{v^2}{r} = \omega^2 r$$

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For a rigid object, the force that supplies this acceleration will be some internal forces between the mass at the rotating point and the other masses in the object. Those are the forces that hold the object together.

Torque

Torque is a measure of force-causing-rotation.

It is not a force, but it is related. It depends on a force vector and its point of application relative to an axis of rotation.

Torque is given by:

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

That is: the cross product between

- a vector $\vec{\mathbf{r}}$, the displacement of the point of application of the force from the axis of rotation, and
- the force vector $\vec{\mathbf{F}}$

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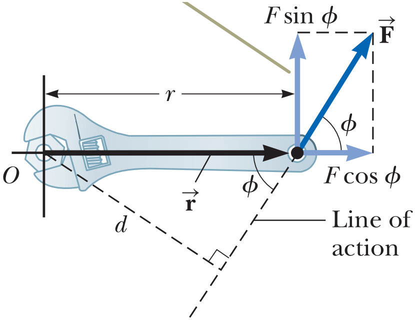
Units: N m   Newton-meters. These are not Joules!

Torque



$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = rF \sin \phi \, \hat{\mathbf{n}}$$

where ϕ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$, as determined by the right-hand rule.

Vectors Properties and Operations

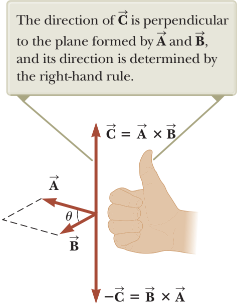
Multiplication by a vector:

The Cross Product

Let $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$
$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}$,

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_x B_y - A_y B_x)\hat{\mathbf{k}}$$

The output of this operation is a vector.



Equivalently,

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = AB \sin\theta \ \hat{\mathbf{n}}_{\mathbf{AB}}$$

where $\hat{\mathbf{n}}_{\mathbf{AB}}$ is a unit vector perpendicular to $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Vectors Properties and Operations

(See page 336 in Serway and Jewett.)

The Cross Product - with $\hat{\mathbf{k}}$ components

In general: $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$
$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$,

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

How do we usually implement this formula?
Via the determinant of a matrix:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector Operations: Cross Product Practice

Try it yourself! Find $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ when:

$$\vec{\mathbf{A}} = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}} \;; \quad \vec{\mathbf{B}} = -1\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

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Now find $\vec{\mathbf{B}} \times \vec{\mathbf{A}}$...
First $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = 22\hat{\mathbf{i}} - 8\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

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Try it yourself! Find $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ when:

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$$\vec{\mathbf{B}} \times \vec{\mathbf{A}} = -22\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Vectors Properties and Operations

(See page 336 in Serway and Jewett.)

The Cross Product - with $\hat{\mathbf{k}}$ components

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = AB \sin\theta \; \hat{\mathbf{n}}_{\mathbf{AB}}$$

Vectors Properties and Operations

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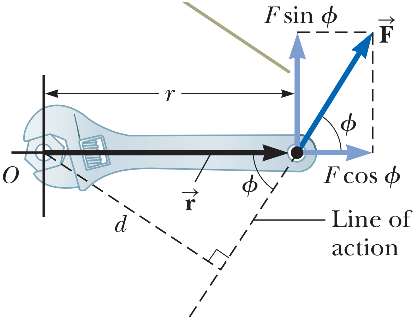
Properties

- The cross product is **not** commutative: $\vec{\mathbf{A}} \times \vec{\mathbf{B}} \neq \vec{\mathbf{B}} \times \vec{\mathbf{A}}$. In fact, it is *anticommutative* because $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -(\vec{\mathbf{B}} \times \vec{\mathbf{A}})$.
- If $\vec{\mathbf{A}} \parallel \vec{\mathbf{B}}$, $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = 0$.
- If $\vec{\mathbf{A}} \perp \vec{\mathbf{B}}$, $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = AB\ \hat{\mathbf{n}}_{\mathbf{AB}}$.

Torque

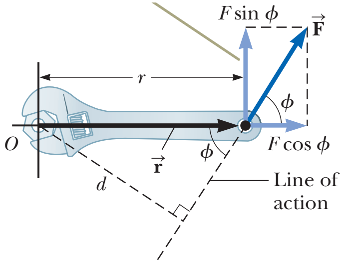


$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = rF \sin \phi \, \hat{\mathbf{n}}$$

where ϕ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$, as determined by the right-hand rule.

Torque

Diagram also illustrates two points of view about torque:



$$\vec{\boldsymbol{\tau}} = r(F \sin \phi)\,\hat{\mathbf{n}}$$

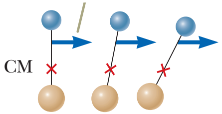
or

$$\vec{\boldsymbol{\tau}} = (r \sin \phi)F\,\hat{\mathbf{n}}$$

In the diagram, the distance $d = r \sin \phi$ and is called the "moment arm" or "lever arm" of the torque.

Torque

Torque:



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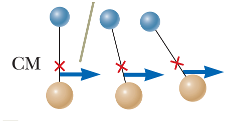
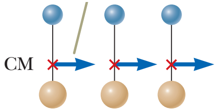
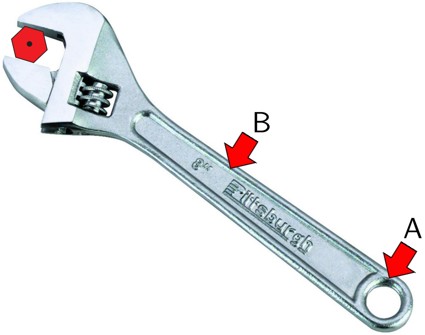


No torque:



Question



A torque is supplied by applying a force at point A. To produce the same torque, the force applied at point B must be:

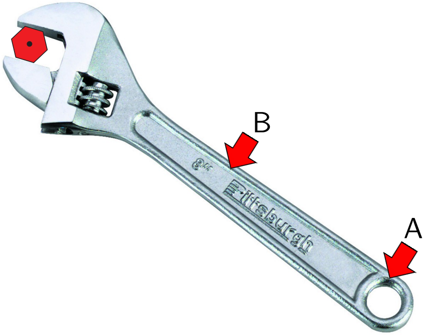
(A) greater

(B) less

(C) the same

[1] Image from Harbor Freight Tools, www.harborfreight.com

Question



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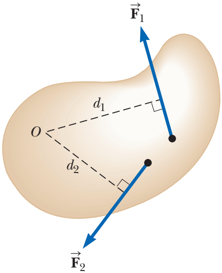
Net Torque

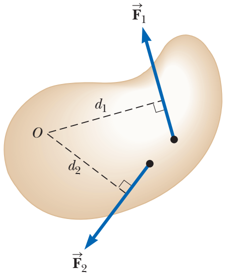
Object that can rotate about an axis at O:



There are two forces acting, but the two torques produced, $\vec{\boldsymbol{\tau}}_1$ and $\vec{\boldsymbol{\tau}}_2$ point in opposite directions.

$\vec{\boldsymbol{\tau}}_1$ would produce a counterclockwise rotation
$\vec{\boldsymbol{\tau}}_2$ would produce a clockwise rotation

Net Torque



The *net torque* is the sum of the torques acting on the object:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \sum_i \vec{\boldsymbol{\tau}}_i$$

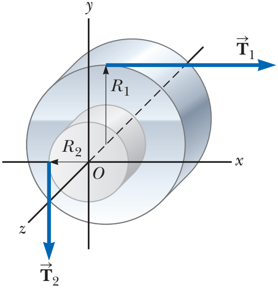
In this case, with $\hat{\mathbf{n}}$ pointing out of the slide:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \vec{\boldsymbol{\tau}}_1 + \vec{\boldsymbol{\tau}}_2 = (d_1 F_1 - d_2 F_2)\hat{\mathbf{n}}$$

Example 10.3 - Net Torque on a Cylinder

A one-piece cylinder is shaped as shown, with a core section protruding from the larger drum. The cylinder is free to rotate about the central z axis shown in the drawing. A rope wrapped around the drum, which has radius R_1, exerts a force T_1 to the right on the cylinder. A rope wrapped around the core, which has radius R_2, exerts a force T_2 downward on the cylinder.

What is the net torque acting on the cylinder about the rotation axis (which is the z axis)?



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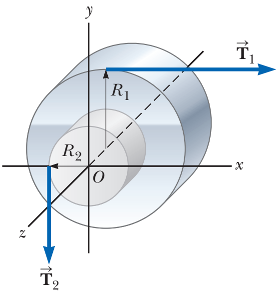
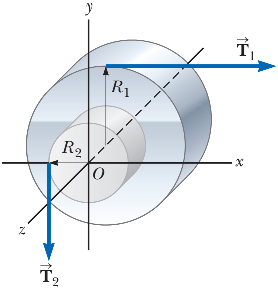


First: Find an expression for the net torque acting on the cylinder about the rotation axis.

Second: Let $T_1 = 5.0$ N, $R_1 = 1.0$ m, $T_2 = 15$ N, and $R_2 = 0.50$ m. What is the net torque? Which way is the rotation?

Example 10.3 - Net Torque on a Cylinder



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$$\vec{\boldsymbol{\tau}}_{\text{net}} = 2.5 \text{ Nm, counter-clockwise } \text{ (or } \hat{\mathbf{k}})$$

Summary

- rotation
- rotational kinematics
- torque

3rd Assignment will be posted later today, watch for an email.

(Uncollected) Homework Serway & Jewett,

- Read ahead in Chapter 10.
- prev: Ch 10, onward from page 288. Probs: 3, 7, 11, 15, 17, 19, 21, 25
- new: Ch 10, Probs: 27 (net torque)