Rotation
Torque
Moment of Inertia

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De Anza College

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Last time

- rotational quantities
- torque
- the cross product

Overview

- net torque
- Newton's second law for rotation
- moment of inertia
- calculating moments of inertia

Quick review of Vector Expressions

Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ be (non-null) vectors.

Could this possibly be a valid equation?

$$\vec{\mathbf{a}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$$

(A) yes

(B) no

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$$m = \vec{\mathbf{b}} \times \vec{\mathbf{c}}$$

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- $\vec{\mathbf{a}} \neq 0$ and
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for *any* value of n. Could this possibly be a true equation?

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Torque

Torque is a measure of force-causing-rotation.

It is not a force, but it is related. It depends on a force vector and its point of application relative to an axis of rotation.

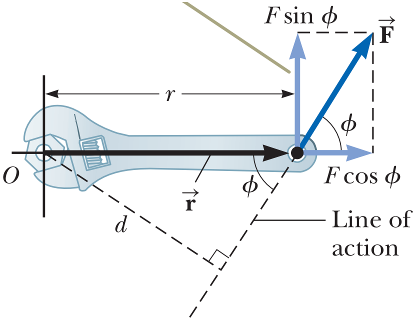
Torque is given by:

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

That is: the cross product between

- a vector $\vec{\mathbf{r}}$, the displacement of the point of application of the force from the axis of rotation, and
- an the force vector $\vec{\mathbf{F}}$

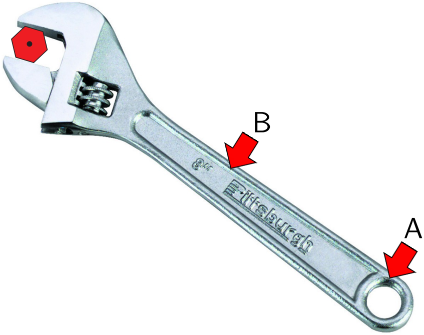
Torque



$$\boldsymbol{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = rF \sin \phi \, \hat{\mathbf{n}}$$

where ϕ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$, as determined by the right-hand rule.

Question



A torque is supplied by applying a force at point A. To produce the same torque, the force applied at point B must be:

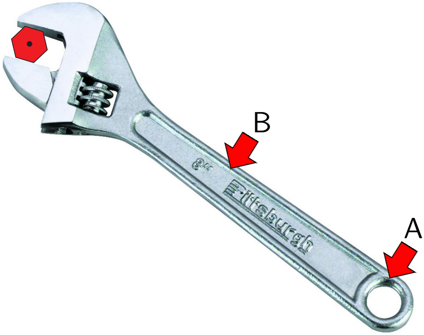
(A) greater

(B) less

(C) the same

[1]Image from Harbor Freight Tools, www.harborfreight.com

Question



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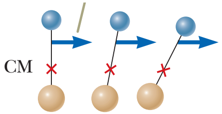

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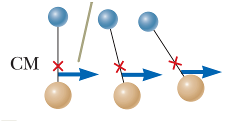
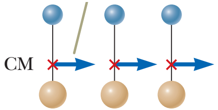
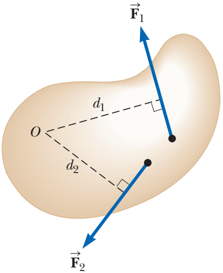


No torque:



Net Torque

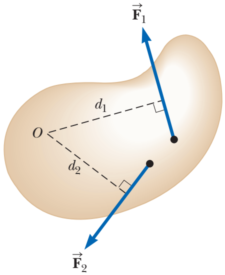
Object that can rotate about an axis at O:



There are two forces acting, but the two torques produced, $\vec{\boldsymbol{\tau}}_1$ and $\vec{\boldsymbol{\tau}}_2$ point in opposite directions.

$\vec{\boldsymbol{\tau}}_1$ would produce a counterclockwise rotation
$\vec{\boldsymbol{\tau}}_2$ would produce a clockwise rotation

Net Torque



The *net torque* is the sum of the torques acting on the object:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \sum_i \vec{\boldsymbol{\tau}}_i$$

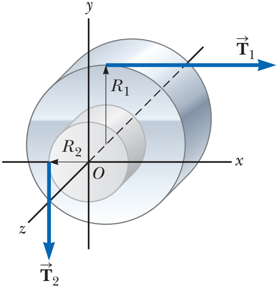
In this case, with $\hat{\mathbf{n}}$ pointing out of the slide:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \vec{\boldsymbol{\tau}}_1 + \vec{\boldsymbol{\tau}}_2 = (d_1 F_1 - d_2 F_2)\hat{\mathbf{n}}$$

Example 10.3 - Net Torque on a Cylinder

A one-piece cylinder is shaped as shown, with a core section protruding from the larger drum. The cylinder is free to rotate about the central z axis shown in the drawing. A rope wrapped around the drum, which has radius R_1, exerts a force T_1 to the right on the cylinder. A rope wrapped around the core, which has radius R_2, exerts a force T_2 downward on the cylinder.

What is the net torque acting on the cylinder about the rotation axis (which is the z axis)?



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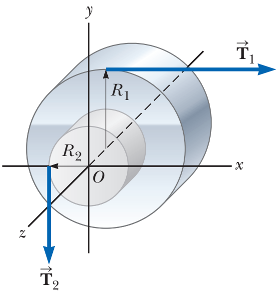

First: Find an expression for the net torque acting on the cylinder about the rotation axis.

Second: Let $T_1 = 5.0$ N, $R_1 = 1.0$ m, $T_2 = 15$ N, and $R_2 = 0.50$ m. What is the net torque? Which way is the rotation?

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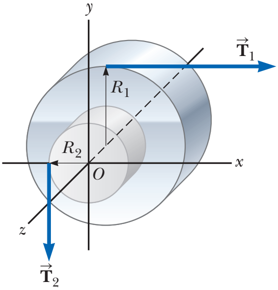

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$$\vec{\boldsymbol{\tau}}_{\text{net}} = 2.5 \text{ Nm, counter-clockwise } \ (\text{or } \hat{\mathbf{k}})$$

Rotational Version of Newton's Second Law

Tangential components of forces give rise to torques.

They also cause tangential accelerations. Consider the tangential component of the net force, $F_{\text{net},t}$:

$$F_{\text{net},t} = m\, a_t$$

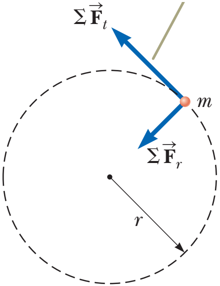
from Newton's second law.

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{\text{net}} = r\, F_{\text{net},t}\, \hat{\mathbf{n}}$$

Now let's specifically consider the case of a single particle, mass m, at a fixed radius r.

Rotational Version of Newton's Second Law

A single particle, mass m, at a fixed radius r.



For such a particle, $F_{\text{net},t} = ma_t$

$$
\begin{aligned}
\vec{\boldsymbol{\tau}}_{\text{net}} &= r\,F_{\text{net},t}\,\hat{\mathbf{n}} \\
&= r\,m\,a_t\,\hat{\mathbf{n}} \\
&= r\,m\,(\vec{\boldsymbol{\alpha}}r) \\
&= (mr^2)\,\vec{\boldsymbol{\alpha}}
\end{aligned}
$$

Rotational Version of Newton's Second Law

(mr^2) is just some constant for this particle and this axis of rotation.

Let this constant be (scalar) $I = mr^2$.

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Replacing the constant quantity in our expression for $\vec{\boldsymbol{\tau}}_{\text{net}}$:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = I\vec{\boldsymbol{\alpha}}$$

Rotational Version of Newton's Second Law

Compare!

$$\vec{\boldsymbol{\tau}}_{\text{net}} = I\,\vec{\boldsymbol{\alpha}}$$

$$\vec{\mathbf{F}}_{\text{net}} = m\,\vec{\mathbf{a}}$$

Now the moment of inertia, I, stands in for the inertial mass, m.

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The moment of inertia measures the rotational inertia of an object (how hard is it to rotate an object?), just as mass is a measure of inertia.

Moment of Inertia

We just found that for a single particle, mass m, radius r,

$$I = mr^2$$

For an **extended object** with mass distributed over varying distances from the rotational axis, each particle of mass m_i experiences a torque:

$$\vec{\boldsymbol{\tau}}_i = m_i r_i^2 \vec{\boldsymbol{\alpha}}$$

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For an **extended object** with mass distributed over varying distances from the rotational axis, each particle of mass m_i experiences a torque:

$$\vec{\boldsymbol{\tau}}_i = m_i r_i^2 \vec{\boldsymbol{\alpha}}$$

And we sum over these torques to get the net torque.

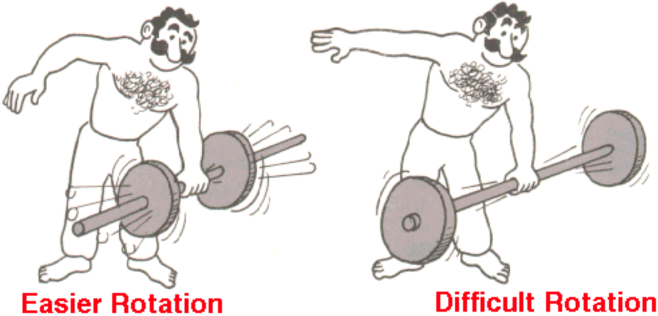
$$\overrightarrow{\boldsymbol{\tau}_{\text{net}}} = \sum_i m_i r_i^2 \vec{\boldsymbol{\alpha}}$$

All particle in the object rotate together, so for an object made of a collection of particles, masses m_i at radiuses r_i:

$$I = \sum_i m_i r_i^2$$

Moment of Inertia

If the object's mass is far from the point of rotation, more torque is needed to rotate the object (with some angular acceleration).



The barbell on the right has a greater moment of inertia.

[1]Diagram from Dr. Hunter's page at http://biomech.byu.edu (by Hewitt?)

Moment of Inertia

And we sum over these torques to get the net torque. So, for a collection of particles, masses m_i at radiuses r_i:

$$I = \sum_i m_i r_i^2$$

For a continuous distribution of mass, we must integrate over each small mass Δm:

$$I = \int r^2 \, \mathrm{dm}$$

Moment of Inertia

For a continuous distribution of mass, we must integrate over each small mass Δm:

$$I = \int r^2 \, \mathrm{dm}$$

For an object of density ρ:

$$I = \int r^2 \, \rho \, \mathrm{dV}$$

If ρ varies with position, it must stay inside the integral.

Moment of Inertia

Important caveat: Moment of inertia depends on the object's mass, shape, and the axis of rotation.

A single object will have **different moments of inertia** for **different axes** of rotation.

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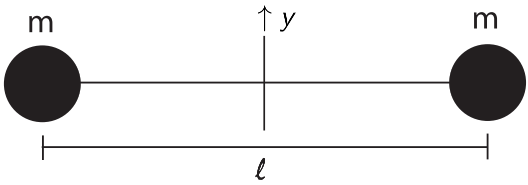
A single object will have **different moments of inertia** for **different axes** of rotation.

Also notice that these integrals and sums are similar to the expression for the center-of-mass, but for I we have r^2 and we do not divide by the total mass.

Units: kg m^2

Example: Calculating Moment of Inertia

Calculate the moment of inertia of two equal point-like masses connected by a light rod, length ℓ, rotating about the y-axis through the center of mass.



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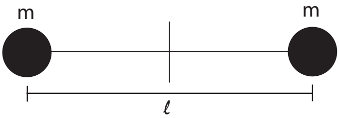

Moment of inertia:

$$
\begin{aligned}
I &= \sum_i m_i x_i^2 \\
&= m\left(-\frac{\ell}{2}\right)^2 + m\left(\frac{\ell}{2}\right)^2
\end{aligned}
$$

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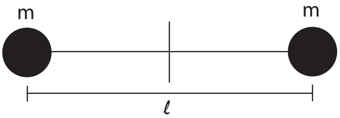

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\begin{aligned}
I &= \sum_i m_i x_i^2 \\
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&= \frac{m\ell^2}{2}
\end{aligned}
$$

Summary

- net torque
- Newton's second law for rotation
- moments of inertia

3rd Assignment due Friday.

(Uncollected) Homework Serway & Jewett,

- Ch 10, Probs: 27 (net torque), 29 (and ang. acc.), 40, 45(a) (moment of interia)
- Look at examples **10.4**, and **10.11**