Rotation Moment of Inertia

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Last time

- net torque
- Newton’s second law for rotation
- moments of inertia
Overview

• calculating moments of inertia

• the parallel axis theorem

• applications of moments of inertia
Rotational Version of Newton’s Second Law

Compare!

\[ \overrightarrow{τ}_{\text{net}} = I \overrightarrow{α} \]
\[ \overrightarrow{F}_{\text{net}} = m \overrightarrow{a} \]

Now the moment of inertia, \( I \), stands in for the inertial mass, \( m \).
Rotational Version of Newton’s Second Law

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\[ \vec{F}_{\text{net}} = m \vec{a} \]

Now the moment of inertia, \( I \), stands in for the inertial mass, \( m \).

The moment of inertia measures the rotational inertia of an object (how hard is it to rotate an object?), just as mass is a measure of inertia.
Moment of Inertia

We just found that for a single particle, mass $m$, radius $r$,

$$ I = mr^2 $$

For an **extended object** with mass distributed over varying distances from the rotational axis, each particle of mass $m_i$ experiences a torque:

$$ \tau_i = m_i r_i^2 \vec{\alpha} $$
Moment of Inertia

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For an extended object with mass distributed over varying distances from the rotational axis, each particle of mass \( m_i \) experiences a torque:

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And we sum over these torques to get the net torque.

\[ \tau_{\text{net}} = \sum_i m_i r_i^2 \hat{\alpha} \]

All particle in the object rotate together, so for an object made of a collection of particles, masses \( m_i \) at radiiuses \( r_i \):

\[ I = \sum_i m_i r_i^2 \]
Moment of Inertia

If the object’s mass is far from the point of rotation, more torque is needed to rotate the object (with some angular acceleration).

Easier Rotation

Difficult Rotation

The barbell on the right has a greater moment of inertia.

\(^1\)Diagram from Dr. Hunter’s page at http://biomech.byu.edu (by Hewitt?)
Moment of Inertia

And we sum over these torques to get the net torque. So, for a collection of particles, masses $m_i$ at radiuses $r_i$:

$$I = \sum_i m_i r_i^2$$

For a continuous distribution of mass, we must integrate over each small mass $\Delta m$:

$$I = \int r^2 \, dm$$
Moment of Inertia

For a continuous distribution of mass, we must integrate over each small mass $\Delta m$:

$$I = \int r^2 \, dm$$

For an object of density $\rho$:

$$I = \int r^2 \rho \, dV$$

If $\rho$ varies with position, it must stay inside the integral.
Important caveat: Moment of inertia depends on the object’s mass, shape, and the axis of rotation.

A single object will have **different moments of inertia** for **different axes** of rotation.
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A single object will have different moments of inertia for different axes of rotation.

Also notice that these integrals and sums are similar to the expression for the center-of-mass, but for $I$ we have $r^2$ and we do not divide by the total mass.

Units: kg m$^2$
Example: Calculating Moment of Inertia

Calculate the moment of inertia of two equal point-like masses connected by a light rod, length $\ell$, rotating about the center of mass.
Example: Calculating Moment of Inertia

Calculate the moment of inertia of two equal point-like masses connected by a light rod, length $\ell$, rotating about the center of mass.

Let the CM be the origin. It will be in the center of the rod.
Example: Calculating Moment of Inertia

Moment of inertia:

\[ I = \sum_i m_i x_i^2 \]

\[ = m \left( -\frac{\ell}{2} \right)^2 + m \left( \frac{\ell}{2} \right)^2 \]
Example: Calculating Moment of Inertia

Moment of inertia:

\[ I = \sum_i m_i x_i^2 \]

\[ = m \left( -\frac{\ell}{2} \right)^2 + m \left( \frac{\ell}{2} \right)^2 \]

\[ = m\ell^2 \]

\[ = \frac{m\ell^2}{2} \]
Calculating Moment of Inertia of a Uniform Rod

(Example 10.7)

Moment of inertia of a uniform thin rod of length $L$ and mass $M$ about an axis perpendicular to the rod (the $y'$ axis) and passing through its center of mass.
Calculating Moment of Inertia of a Uniform Rod

Rod is uniform: let $\lambda = \frac{M}{L}$ be the mass per unit length (density).

$$I_y' = \int r^2 \, dm$$
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I_y' = \int r^2 \, dm = \int_{-L/2}^{L/2} (x')^2 \lambda \, dx'
\]
Calculating Moment of Inertia of a Uniform Rod

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$$I_y' = \int r^2 \, dm$$

$$= \int_{-L/2}^{L/2} (x')^2 \lambda \, dx'$$

$$= \lambda \left[ \frac{(x')^3}{3} \right]_{-L/2}^{L/2}$$

$$= \frac{M}{L} \left[ \frac{L^3}{24} + \frac{L^3}{24} \right]$$

$$= \frac{1}{12} ML^2$$
Calculating Moment of Inertia of a Uniform Rod

Moment of inertia of a uniform thin rod of length \( L \) and mass \( M \) about an axis perpendicular to the rod (the \( y' \) axis) and passing through its center of mass.

\[
I_{y'} = \frac{1}{12} M L^2
\]
Suppose you know the moment of inertia of an object about an axis through its center of mass.

Let that axis be the z-axis. (The coordinates of the center of mass are (0, 0, 0).)
A Trick: The Parallel Axis Theorem

Suppose you know the moment of inertia of an object about an axis through its center of mass.

Let that axis be the $z$-axis. (The coordinates of the center of mass are $(0, 0, 0)$.)

Suppose you want to know the moment of inertia about a different axis, parallel to the first one. Let the $(x, y)$ coordinates of this new axis $z'$ be $(a, b)$. 
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Let that axis be the z-axis. (The coordinates of the center of mass are \((0, 0, 0)\).)

Suppose you want to know the moment of inertia about a different axis, parallel to the first one. Let the \((x, y)\) coordinates of this new axis \(z'\) be \((a, b)\).

Then we must integrate:

\[
I_{z'} = \int (r')^2 \, dm
\]

where \(r'\) is the distance of the mass increment \(dm\) from the new axis, \(z'\).
The Parallel Axis Theorem

The result we will get.

For an axis through the center of mass and any parallel axis through some other point:

\[ I_\parallel = I_{CM} + M D^2 \]

where \( D = \sqrt{a^2 + b^2} \) is the distance from the axis through the center of mass to the new axis.
The Parallel Axis Theorem

Distance formula: \((r')^2 = (x - a)^2 + (y - b)^2\).
The Parallel Axis Theorem

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\[
I_z' = \int (r')^2 \, dm \\
= \int (x - a)^2 + (y - b)^2 \, dm \\
= \int x^2 \, dm - 2a \int x \, dm + \int a^2 \, dm \\
+ \int y^2 \, dm - 2b \int y \, dm + \int b^2 \, dm
\]
The Parallel Axis Theorem

But: \[ x_{CM} = \frac{1}{M} \int x \, dm = 0 \Rightarrow \int x \, dm = 0. \] Also for \( y \).
The Parallel Axis Theorem

But: $x_{CM} = \frac{1}{M} \int x \, dm = 0 \Rightarrow \int x \, dm = 0$. Also for $y$.

$$I_z' = \int x^2 \, dm - 2a \int x \, dm + \int a^2 \, dm$$

$$+ \int y^2 \, dm - 2b \int y \, dm + \int b^2 \, dm$$

where $D = \sqrt{a^2 + b^2}$ is the distance from from the origin to the new axis.
The Parallel Axis Theorem

But: $x_{CM} = \frac{1}{M} \int x \, dm = 0 \Rightarrow \int x \, dm = 0$. Also for $y$.

\[
I_{z'} = \int x^2 \, dm - 2a \int x \, dm + \int a^2 \, dm + \int y^2 \, dm - 2b \int y \, dm + \int b^2 \, dm
\]

where $D = \sqrt{a^2 + b^2}$ is the distance from the origin to the new axis.
The Parallel Axis Theorem

But: $x_{CM} = \frac{1}{M} \int x \, dm = 0 \Rightarrow \int x \, dm = 0$. Also for $y$.

\[
I_{z'} = \int x^2 \, dm - 2a \int x \, dm + \int a^2 \, dm + \int y^2 \, dm - 2b \int y \, dm + \int b^2 \, dm
\]

\[
= \int (x^2 + y^2) \, dm + \int (a^2 + b^2) \, dm
\]

where $D = \sqrt{a^2 + b^2}$ is the distance from the origin to the new axis.
The Parallel Axis Theorem

But: $x_{CM} = \frac{1}{M} \int x \, dm = 0 \implies \int x \, dm = 0$. Also for $y$.

\[ I_z' = \int x^2 \, dm - 2a \int x \, dm + \int a^2 \, dm \]

\[ + \int y^2 \, dm - 2b \int y \, dm + \int b^2 \, dm \]

\[ = \int (x^2 + y^2) \, dm + \int (a^2 + b^2) \, dm \]

\[ = I_z + M D^2 \]

where $D = \sqrt{a^2 + b^2}$ is the distance from from the origin to the new axis.
The Parallel Axis Theorem

\[ I_{||} = I_{CM} + MD^2 \]

where \( D = \sqrt{a^2 + b^2} \) is the distance from from the origin to the new axis.
Question

We found that the moment of inertia of a uniform rod about its midpoint was \( I = \frac{1}{12} ML^2 \).

What is the moment of inertia of the same rod about the and axis through an endpoint of the rod?

(A) \( \frac{1}{3} ML^2 \)
(B) \( \frac{1}{2} ML^2 \)
(C) \( \frac{7}{12} ML^2 \)
(D) \( \frac{13}{12} ML^2 \)
Summary

- moments of inertia
- the parallel axis theorem

Next test Thursday, Mar 14.

(Uncollected) Homework Serway & Jewett,

- Ch 10, onward from page 288. Probs: 29, 39, 43, 45(a)