Rotation
Moment of Inertia

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Last time

- net torque
- Newton's second law for rotation
- moments of inertia

Overview

- calculating moments of inertia
- the parallel axis theorem
- applications of moments of inertia

Rotational Version of Newton's Second Law

Compare!

$$\vec{\boldsymbol{\tau}}_{\text{net}} = I\,\vec{\boldsymbol{\alpha}}$$

$$\vec{\mathbf{F}}_{\text{net}} = m\,\vec{\mathbf{a}}$$

Now the moment of inertia, I, stands in for the inertial mass, m.

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Compare!

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Now the moment of inertia, I, stands in for the inertial mass, m.

The moment of inertia measures the rotational inertia of an object (how hard is it to rotate an object?), just as mass is a measure of inertia.

Moment of Inertia

And we sum over these torques to get the net torque. So, for a collection of particles, masses m_i at radiuses r_i:

$$I = \sum_i m_i r_i^2$$

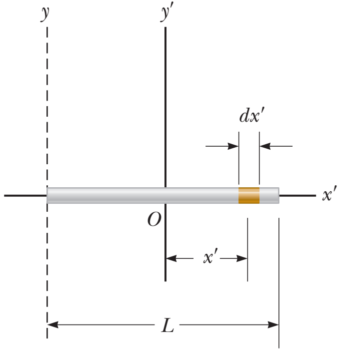
For a continuous distribution of mass, we must integrate over each small mass Δm:

$$I = \int r^2 \, \mathrm{dm}$$

Calculating Moment of Inertia of a Uniform Rod

(Example 10.7)

Moment of inertia of a uniform thin rod of length L and mass M about an axis perpendicular to the rod (the y' axis) and passing through its center of mass.



Calculating Moment of Inertia of a Uniform Rod

Rod is uniform: let $\lambda = \frac{M}{L}$ be the mass per unit length (density).

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&= \int_{-L/2}^{L/2} (x')^2 \, \lambda \, \mathrm{d}x' \\
&= \lambda \left[\frac{(x')^3}{3} \right]_{-L/2}^{L/2} \\
&= \frac{M}{L} \left[\frac{L^3}{24} + \frac{L^3}{24} \right] \\
&= \frac{1}{12} M L^2
\end{aligned}
$$

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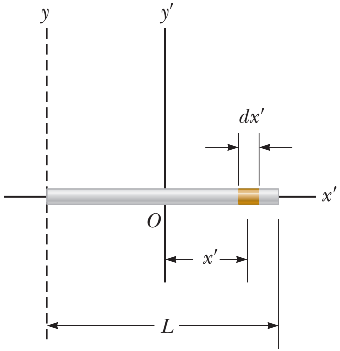

$$I_{y'} = \frac{1}{12} M L^2$$

A Trick: The Parallel Axis Theorem

Suppose you know the moment of inertia of an object about an axis through its center of mass.

Let that axis be the z-axis. (The coordinates of the center of mass are $(0, 0, 0)$.)

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Then we must integrate:

$$I_{z'} = \int (r')^2 \, \mathrm{dm}$$

where r' is the distance of the mass increment dm from the new axis, z'.

The Parallel Axis Theorem

The result we will get.

For an axis through the center of mass and any parallel axis through some other point:

$$I_{\parallel} \;=\; I_{\mathsf{CM}} + M\,D^2$$

where $D = \sqrt{a^2 + b^2}$ is the distance from from the axis through the center of mass to the new axis.

The Parallel Axis Theorem

Distance formula: $(r')^2 = (x-a)^2 + (y-b)^2$.

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\begin{aligned}
I_{z'} &= \int (r')^2 \, \mathrm{dm} \\
&= \int (x-a)^2 + (y-b)^2 \, \mathrm{dm} \\
&= \int x^2 \, \mathrm{dm} - 2a \int x \, \mathrm{dm} + \int a^2 \, \mathrm{dm} \\
&\quad + \int y^2 \, \mathrm{dm} - 2b \int y \, \mathrm{dm} + \int b^2 \, \mathrm{dm}
\end{aligned}
$$

The Parallel Axis Theorem

But: $x_{\mathrm{CM}} = \frac{1}{M}\int x \, \mathrm{dm} = 0 \quad \Rightarrow \quad \int x \, \mathrm{dm} = 0$. Also for y.

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&\quad + \int y^2\,\text{dm} - 2b\cancelto{0}{\int y\,\text{dm}} + \int b^2\,\text{dm} \\
&= \int (x^2 + y^2)\,\text{dm} + \int (a^2 + b^2)\,\text{dm}
\end{aligned}
$$

The Parallel Axis Theorem

But: $x_{CM} = \frac{1}{M}\int x\,\mathrm{dm} = 0 \quad \Rightarrow \quad \int x\,\mathrm{dm} = 0$. Also for y.

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I_{z'} &= \int x^2\,\mathrm{dm} - 2a\cancelto{0}{\int x\,\mathrm{dm}} + \int a^2\,\mathrm{dm} \\
&\quad + \int y^2\,\mathrm{dm} - 2b\cancelto{0}{\int y\,\mathrm{dm}} + \int b^2\,\mathrm{dm} \\
&= \int (x^2 + y^2)\,\mathrm{dm} + \int (a^2 + b^2)\,\mathrm{dm} \\
&= I_z + M\,D^2
\end{aligned}$$

where $D = \sqrt{a^2 + b^2}$ is the distance from from the origin to the new axis.

The Parallel Axis Theorem

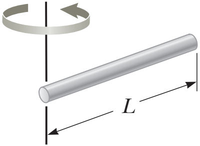
$$I_{\parallel} = I_{\mathrm{CM}} + M\,D^2$$

where $D = \sqrt{a^2 + b^2}$ is the distance from from the origin to the new axis.

Question

We found that the moment of inertia of a uniform rod about its midpoint was $I = \frac{1}{12}ML^2$.

What is the moment of inertia of the same rod about the and axis through an endpoint of the rod?



(A) $\frac{1}{3}ML^2$

(B) $\frac{1}{2}ML^2$

(C) $\frac{7}{12}ML^2$

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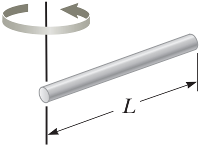

(A) $\frac{1}{3}ML^2$ $\leftarrow$

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(C) $\frac{7}{12}ML^2$

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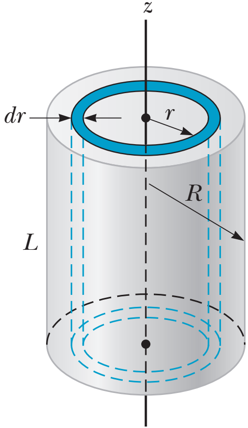
Calculating Moment of Inertia of a Uniform Cylinder

(Example 10.8)

A uniform solid cylinder has a radius R, mass M, and length L. Calculate its moment of inertia about its central axis (the z axis).



Calculating Moment of Inertia of a Uniform Cylinder

We will again need to evaluate $I = \int r^2 \, \mathrm{dm}$. Let $\rho = \frac{M}{V}$, so that $I = \rho \int r^2 \, \mathrm{dV}$.

Calculating Moment of Inertia of a Uniform Cylinder

We will again need to evaluate $I = \int r^2 \, dm$. Let $\rho = \frac{M}{V}$, so that $I = \rho \int r^2 \, dV$.

We will do this by summing up cylindrical shells.

What is the surface area of a cylinder of height L (without the circular ends)?

Calculating Moment of Inertia of a Uniform Cylinder

We will again need to evaluate $I = \int r^2 \, \mathrm{dm}$. Let $\rho = \frac{M}{V}$, so that $I = \rho \int r^2 \, \mathrm{dV}$.

We will do this by summing up cylindrical shells.

What is the surface area of a cylinder of height L (without the circular ends)? $A = 2\pi r L$

Then the volume of a cylindrical shell of thickness dr is

$$\mathrm{dV} = 2\pi r L \, \mathrm{dr}$$

(The volume of the entire cylinder, $V = \pi R^2 L$.)

Calculating Moment of Inertia of a Uniform Cylinder

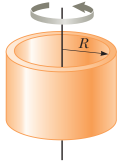
$$
\begin{aligned}
I_z &= \int_0^R r^2 (\rho 2\pi r L \, \mathrm{dr}) \\
I_z &= 2\pi\rho L \int_0^R r^3 \, \mathrm{dr}
\end{aligned}
$$

Calculating Moment of Inertia of a Uniform Cylinder

$$
\begin{aligned}
I_z &= \int_0^R r^2(\rho 2\pi r L\,\mathrm{dr}) \\
I_z &= 2\pi\rho L \int_0^R r^3\,\mathrm{dr} \\
&= 2\pi\rho L \left[\frac{r^4}{4}\right]_0^R \\
&= 2\pi\rho L \left[\frac{R^4}{4}\right] \\
&= \frac{1}{2}\pi L \left(\frac{M}{\pi R^2 L}\right) R^4 \\
&= \frac{1}{2} M R^2
\end{aligned}
$$

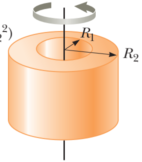
Hoop or thin cylindrical shell

$I_{CM} = MR^2$



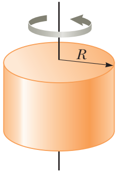
Hollow cylinder

$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



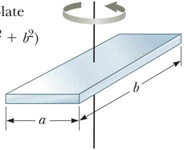
Solid cylinder or disk

$I_{CM} = \frac{1}{2}MR^2$



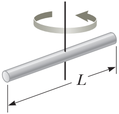
Rectangular plate

$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



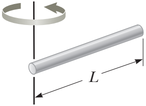
Long, thin rod with rotation axis through center

$I_{CM} = \frac{1}{12}ML^2$



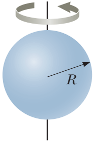
Long, thin rod with rotation axis through end

$I = \frac{1}{3}ML^2$



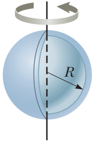
Solid sphere

$I_{CM} = \frac{2}{5}MR^2$



Thin spherical shell

$I_{CM} = \frac{2}{3}MR^2$



Summary

- moments of inertia
- the parallel axis theorem

Quiz Thursday.

3rd Assignment due Friday.

(Uncollected) Homework Serway & Jewett,

- Ch 10, onward from page 288. Probs: 39, 43, 46 (try the trick we used with finding CMs - can wait until tomorrow, when we cover KE)
- Look at example **10.6**. (can wait until tomorrow)