Rotation
Moment of Inertia and Applications
Atwood Machine with Massive Pulley
Energy of Rotation

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Last time

- calculating moments of inertia
- the parallel axis theorem

Overview

- applications of moments of inertia
- Atwood machine with massive pulley
- work, kinetic energy, and power of rotation

Applying Moments of Inertia

Now that we can find moments of inertia of various objects, we can use them to calculate angular accelerations from torques, and vice versa.

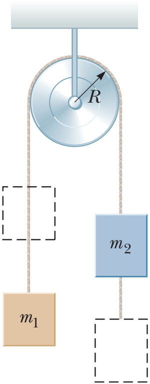
We can solve for the motion of systems with rotating parts.

$$\vec{\boldsymbol{\tau}} = I\vec{\boldsymbol{\alpha}}$$

The Atwood Machine Revisited

Remember that previously we studied the Atwood machine, assuming the pulley was massless.

What if it's not?

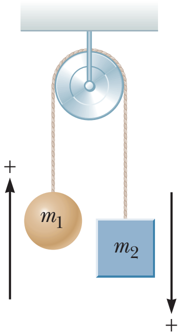


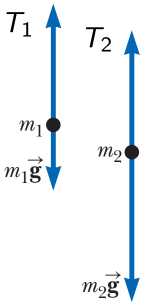
[1]See the slides from lecture 8.

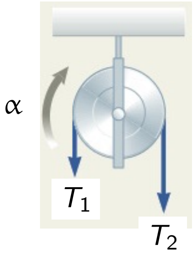
The Atwood Machine Revisited

Suppose the pulley has mass M and we model it as a cylinder, radius R.

Then $I = \frac{1}{2}MR^2$ for the pulley. Torque is needed to accelerate it.







The Atwood Machine Revisited

Find the acceleration of the masses?

The Atwood Machine Revisited

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Forces on object 1:

$$F_{\text{net},1} = m_1 a = T_1 - m_1 g$$

Forces on object 2:

$$F_{\text{net},2} = m_2 a = m_2 g - T_2$$

Torque on pulley:

$$\tau_{\text{net}} = I\alpha = T_2 R - T_1 R$$

The Atwood Machine Revisited

From the torque equation:

$$\begin{aligned} I\alpha &= T_2R - T_1R \\ \frac{I}{R}\alpha &= T_2 - T_1 \end{aligned}$$

The Atwood Machine Revisited

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\left(\frac{I}{R^2} + m_1 + m_2\right) a &= (m_2 - m_1) g
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Putting in $I = \frac{1}{2} M R^2$:

$$a = \frac{(m_2 - m_1) g}{\left(\frac{M}{2} + m_1 + m_2\right)}$$

Rotational Kinetic Energy

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Imagine an object made up of a collection of particles, mass m_i, radius r_i. The kinetic energy or each particle is

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Notice that $I = \sum_i m_i r_i^2$.

Rotational Kinetic Energy

Kinetic energy of a rigid object rotating at an angular speed ω is

$$K = \frac{1}{2} I \omega^2$$

Kinetic Energy of Rotation

Quick Quiz 10.6[1] A section of hollow pipe and a solid cylinder have the same radius, mass, and length. They both rotate about their long central axes with the same angular speed. Which object has the higher rotational kinetic energy?

(A) The hollow pipe does.

(B) The solid cylinder does.

(C) They have the same rotational kinetic energy.

(D) It is impossible to determine.

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Energy

In total then,

$$W_{\text{ext}} = \Delta K_{\text{trans}} + \Delta K_{\text{rot}} + \Delta U + \Delta E_{\text{int}}$$

where

- ΔK_{trans} represents the kinetic energy of the CM motion, and
- ΔK_{rot} is the rotational kinetic energy

Work of Rotation

We can define the work done by a torque $\vec{\tau}$ over a rotation of angular displacement $\overrightarrow{\Delta\theta} = \vec{\theta}_f - \vec{\theta}_i$:

$$W = \int_{\theta_i}^{\theta_f} \vec{\tau} \cdot \mathrm{d}\vec{\theta}$$

This is equivalent to the force integral definition.

Work of Rotation

$$W = \int \vec{\mathbf{F}} \cdot \mathrm{d}\,\vec{\mathbf{s}}$$

Work of Rotation

$$
\begin{aligned}
W &= \int \vec{\mathbf{F}} \cdot \mathrm{d}\,\vec{\mathbf{s}} \\
&= \int F_t \ \mathrm{ds} \\
&= \int F_t r \,\mathrm{d}\theta \qquad (\mathrm{ds} = r\mathrm{d}\theta) \\
&= \int \tau \,\mathrm{d}\theta \qquad (\tau = r\, F_t) \\
&= \int \vec{\boldsymbol{\tau}} \cdot \mathrm{d}\,\vec{\boldsymbol{\theta}}
\end{aligned}
$$

The last line follow because the τ and θ vectors point along the same (fixed) axis.

Power

$$W = \int \tau d\theta$$

implies:

$$\frac{dW}{d\theta} = \tau$$

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\frac{dW}{d\theta} &= \tau \\
\frac{dW}{dt}\frac{dt}{d\theta} &= \tau \\
\frac{dW}{dt} &= \tau \frac{d\theta}{dt}
\end{aligned}$$

Power

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implies:

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\frac{\mathrm{d}W}{\mathrm{d}\theta} &= \tau \\
\frac{\mathrm{dW}}{\mathrm{dt}} \frac{\mathrm{d}t}{\mathrm{d}\theta} &= \tau \\
\frac{\mathrm{dW}}{\mathrm{dt}} &= \tau \frac{\mathrm{d}\theta}{\mathrm{d}t}
\end{aligned}$$

Giving an expression for power:

$$P = \tau\omega$$

Work-Kinetic Energy Theorem

$$
\begin{aligned}
W &= \int \tau \mathrm{d}\theta \\
&= \int (I\alpha) \mathrm{d}\theta
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&= \int I \left(\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\mathrm{d}\theta \\
&= \int I\omega \, \mathrm{d}\omega
\end{aligned}
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&= \int I \left(\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\mathrm{d}\theta \\
&= \int I\omega\,\mathrm{d}\omega \\
&= \Delta\left(\frac{1}{2}I\omega^2\right)
\end{aligned}$$

$$W = \Delta K$$

Example - Moment of Inertia and Rotational KE

Page 328, #44

44. W Rigid rods of negligible mass lying along the y axis connect three particles (Fig. P10.44). The system rotates about the x axis with an angular speed of 2.00 rad/s. Find (a) the moment of inertia about the x axis, (b) the total rotational kinetic energy evaluated from $\frac{1}{2}I\omega^2$, (c) the tangential speed of each particle, and (d) the total kinetic energy evaluated from $\sum \frac{1}{2}m_i v_i^2$. (e) Compare the answers for kinetic energy in parts (a) and (b).

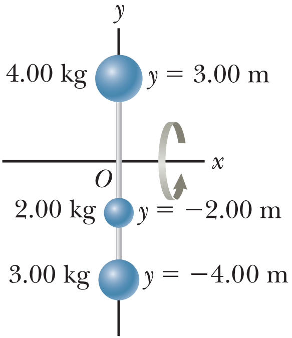

Figure P10.44

Example - Moment of Inertia and Rotational KE

(a) Moment of inertia about x-axis?

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$$K = 184 \text{ J}$$

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(a) Moment of inertia about x-axis?

$$I_x = 92 \text{ kg m}^2$$

(b) & (d) Kinetic energy?

$$K = 184 \text{ J}$$

(c) tangential speeds?

$$v = r\omega$$

Summary

- Atwood machine revisited
- energy of rotation

Assignment 3 due Friday.

Quiz Thursday.

(Uncollected) Homework Serway & Jewett,

- Ch 10, onward from page 234. Probs: 35, 45(b), 46, 51, 53, 55, 57