Rotation
Rolling Motion
Angular Momentum

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Last time

- moment of inertia volume integral example
- applications of moments of inertia
- Atwood machine with massive pulley

Overview

- kinetic energy, work, and power of rotation
- Rolling motion
- Examples

Work-Kinetic Energy Theorem

$$\begin{aligned} W &= \int \tau \mathrm{d}\theta \\ &= \int (I\alpha) \mathrm{d}\theta \end{aligned}$$

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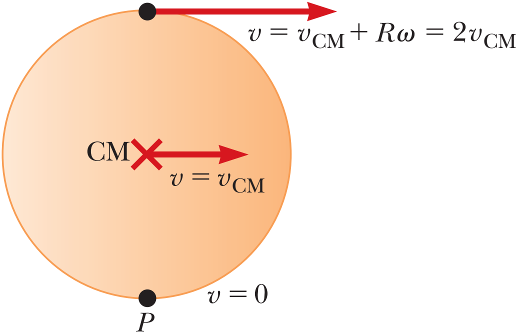
$$W = \Delta K$$

Rolling Motion

A combination of translation and rotation.

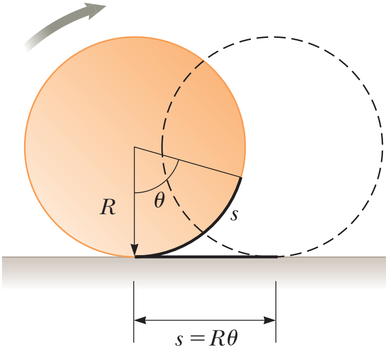


The CM is translated at velocity v_{CM}.

Notice that instantaneously P is the pivot point of the rotation.

Rolling Motion

The pivot point of the rotation changes as different parts of the wheel contact the surface.



$$v_{\mathrm{CM}} = \frac{\mathrm{d}s}{\mathrm{d}t} = R\omega$$

Kinetic Energy of a Rolling Object

The kinetic energy of a rolling object is just the sum of the rotational KE and translational KE. Can see this by considering

just an instantaneous rotation about the point P:

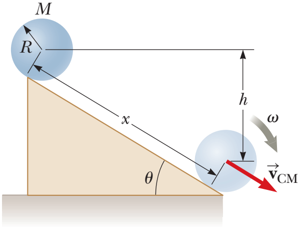
$$K = \frac{1}{2} I_P \omega^2$$

Using the parallel axis theorem: $I_P = I_{\text{CM}} + mR^2$.

$$\begin{aligned} K &= \frac{1}{2}(I_{\text{CM}} + mR^2)\omega^2 \\ &= \frac{1}{2} I_{\text{CM}} \omega^2 + \frac{1}{2} m v_{\text{CM}}^2 \\ &= K_{\text{CM,rot}} + K_{\text{CM,trans}} \end{aligned}$$

Rolling down an incline

A sphere starts from rest at the top of an incline and rolls down. Find the (translational) velocity of the center of mass at the bottom of the incline.



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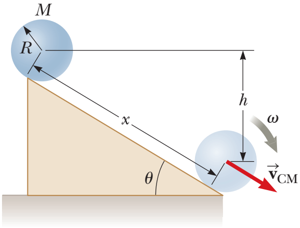

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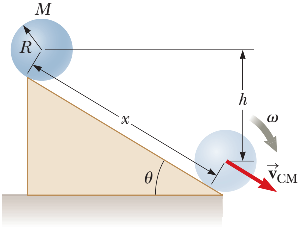

$$\Delta K + \Delta U = 0$$

$$\left(\frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2 - 0\right) + (0 - Mgh) = 0$$

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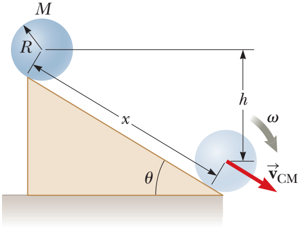

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$$v_{CM} = \sqrt{\frac{2gh}{1 + (I_{CM}/MR^2))}}$$

Question

Quick Quiz 10.7[1] A ball rolls without slipping down incline A, starting from rest. At the same time, a box starts from rest and slides down incline B, which is identical to incline A except that it is frictionless. Which arrives at the bottom first?

(A) The ball arrives first.

(B) The box arrives first.

(C) Both arrive at the same time.

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[1] Serway & Jewett, page 318.

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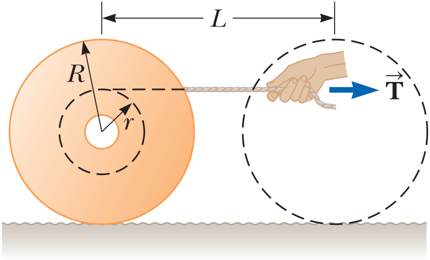
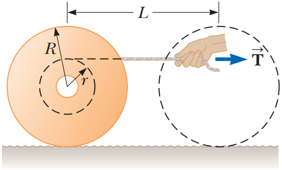
Example 10.14 - Pulling a Spool

A cylindrically symmetric spool of mass m and radius R, and moment of inertia I, sits at rest on a horizontal table with friction. You pull on on a light string wrapped around the axle (radius r) of the spool with a constant horizontal force of magnitude T to the right. As a result, the spool rolls without slipping a distance L along the table with no rolling friction.

Find the final translational speed of the center of mass of the spool.



Example 10.14 - Pulling a Spool



Can use: $W = \Delta K$.

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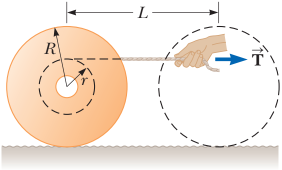


Can use: $W = \Delta K$.

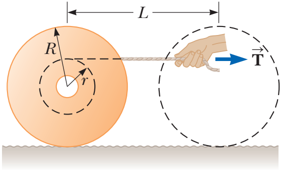
$$W = TL\left(1 + \frac{r}{R}\right)$$

This can be thought of in two ways.

First, the string unrolls from the spool, so the point of application of the force $\vec{F}$ (the hand) moves a distance $L + L\frac{r}{R}$.

Alternatively, there is work done translating the spool: $W = TL$, plus work done rotating the spool: $W = \tau\,\Delta\theta = (rT)\left(\frac{L}{R}\right)$.

Example 10.14 - Pulling a Spool



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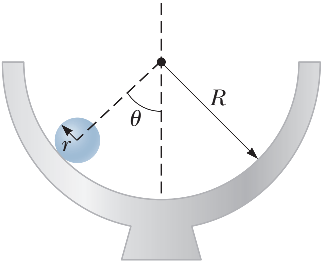
$$v_{CM} = \sqrt{\frac{2TL(1 + r/R)}{m + I/R^2}}$$

Example: Rolling, pg333, # 81

81. A uniform solid sphere of radius r is placed on the inside surface of a hemispherical bowl with radius R. The sphere is released from rest at an angle θ to the vertical and rolls without slipping (Fig. P10.81). Determine the angular speed of the sphere when it reaches the bottom of the bowl.



Example: Rolling, pg333, # 81

Energy conservation:

$$\begin{aligned} \Delta K + \Delta U &= 0 \\ \left(\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 - 0\right) + (0 - mg(R-r)(1-\cos\theta)) &= 0 \end{aligned}$$

$$v = r\omega$$

$$\begin{aligned} \left(\frac{1}{2}(\frac{2}{5}mr^2)\omega^2 + \frac{1}{2}m(r\omega)^2\right) + mg(R-r)(\cos\theta - 1) &= 0 \\ 7mr^2\omega^2 + 10mg(R-r)(\cos\theta - 1) &= 0 \end{aligned}$$

$$\omega = \sqrt{\frac{10g(R-r)(1-\cos\theta)}{7r^2}}$$

Reminder about Force and Torque

Torque is a rotational extension of force, in the sense that it can cause an acceleration / change in momentum:

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

We can write Newton's Second Law in its more general form:

$$\vec{\mathbf{F}}_{\text{net}} = \frac{d\vec{\mathbf{p}}}{dt}$$

This relates **force** to **momentum**.

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Yes!

Angular Momentum

A new quantity, angular momentum:

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

where

- $\vec{\mathbf{r}}$ is the displacement vector of a particle relative to some axis of rotation, and
- $\vec{\mathbf{p}}$ is the momentum of the particle

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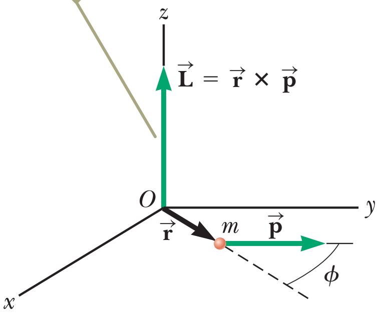
Units: kg m^2 s^{-1}

Angular Momentum

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

The angular momentum $\vec{\mathbf{L}}$ of a particle about an axis is a vector perpendicular to both the particle's position $\vec{\mathbf{r}}$ relative to the axis and its momentum $\vec{\mathbf{p}}$.



Angular Momentum

$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ is a vector equation.

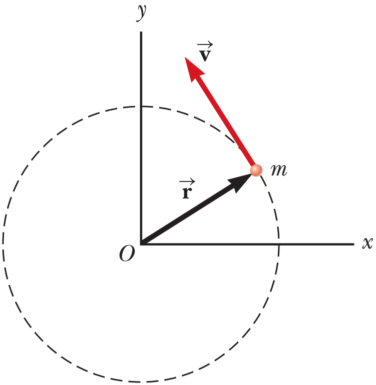
If we only need to know the magnitude of L, then we can use the following expression:

$$L = mvr \sin \phi$$

where we used $p = mv$, and ϕ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$.

Angular Momentum of a Particle in Circular Motion

A particle has mass, m, velocity v, and travels in a circular path of radius r about a point O. What is the magnitude of its angular momentum relative to the axis O?



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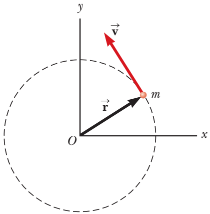

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What is the direction of the angular momentum vector $\vec{\mathbf{L}}$?

(A) $+\hat{\mathbf{k}}$

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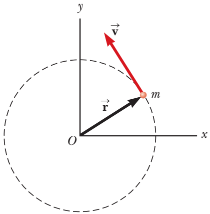

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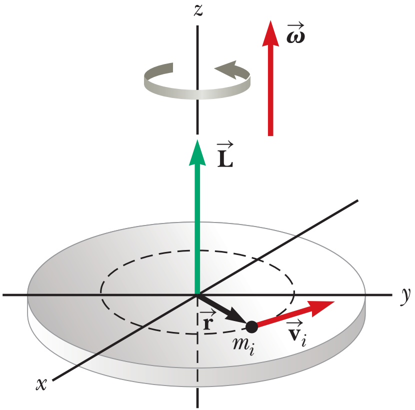
Angular Momentum of Rigid Object



All parts of the object have the same angular velocity $\vec{\omega}$.

Angular Momentum of Rigid Object

Consider a rotating rigid object. Let the z-axis point along the axis of rotation, $\vec{\omega} = \omega\hat{\mathbf{k}}$.

For one particle in the object, mass m_i, at radius r_i:

$$\begin{aligned} \vec{\mathbf{L}} &= m_i v_i r_i \,\hat{\mathbf{k}} \\ &= m_i(\omega r_i) r_i \,\hat{\mathbf{k}} \\ &= m_i r_i^2 \vec{\omega} \end{aligned}$$

For a rigid object made of many particles:

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For a rigid object made of many particles:

$$\begin{aligned} \vec{\mathbf{L}}_{\text{tot}} &= \sum_i \vec{\mathbf{L}}_i \\ &= \left(\sum_i m_i r_i^2 \right) \vec{\omega} \\ &= I\vec{\omega} \end{aligned}$$

Angular Momentum of Rigid Object

For a rigid object:

$$\vec{\mathsf{L}} = I\vec{\omega}$$

where I is the moment of inertia and $\vec{\omega}$ is the angular velocity.

Question

Quick Quiz 11.3[2] A solid sphere and a hollow sphere have the same mass and radius. They are rotating with the same angular speed. Which one has the higher angular momentum?

(A) the solid sphere

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[2]Serway & Jewett, page 343.

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Summary

- energy of rotation
- rolling motion
- examples

3rd Assignment due tomorrow.

(Uncollected) Homework Serway & Jewett,

- Ch 10, rolling motion, CQ 13; Probs: 59, 61, 65.