Rotation
Angular Momentum

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Last time

- rolling motion

Overview

- Definition of angular momentum
- angular momentum of rigid objects
- relation to Newton's 2nd law
- angular impulse

Reminder about Force and Torque

Torque is a rotational extension of force, in the sense that it can cause an acceleration / change in momentum:

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

We can write Newton's Second Law in its more general form:

$$\vec{\mathbf{F}}_{\text{net}} = \frac{d\vec{\mathbf{p}}}{dt}$$

This relates **force** to **momentum**.

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Is there some similar rotational quantity that would relate torque to a change in motion?

Yes!

Angular Momentum

A new quantity, angular momentum:

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

where

- $\vec{\mathbf{r}}$ is the displacement vector of a particle relative to some axis of rotation, and
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Units: kg m^2 s^{-1}

Angular Momentum and Newton's Second Law

This is a more general form of Newton's second law for rotations!

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Compare: $\vec{\mathbf{F}}_{\text{net}} = \frac{d\vec{\mathbf{p}}}{dt}$

Angular Momentum

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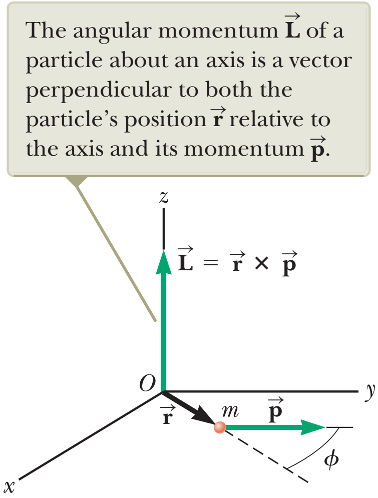


Angular Momentum

$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ is a vector equation.

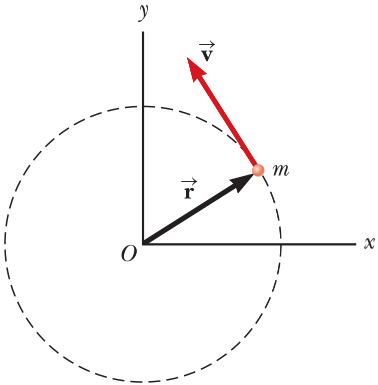
If we only need to know the magnitude of L, then we can use the following expression:

$$L = mvr \sin \phi$$

where we used $p = mv$, and ϕ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$.

Angular Momentum of a Particle in Circular Motion

A particle has mass, m, velocity v, and travels in a circular path of radius r about a point O. What is the magnitude of its angular momentum relative to the axis O?



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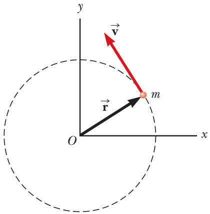


A particle has mass, m, velocity v, and travels in a circular path of radius r about a point O. What is the magnitude of its angular momentum relative to the axis O?

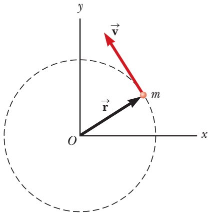
$$L = mrv$$

What is the direction of the angular momentum vector $\vec{\mathbf{L}}$?

(A) $+\hat{\mathbf{k}}$

(B) $-\hat{\mathbf{k}}$

Angular Momentum of a Particle in Circular Motion



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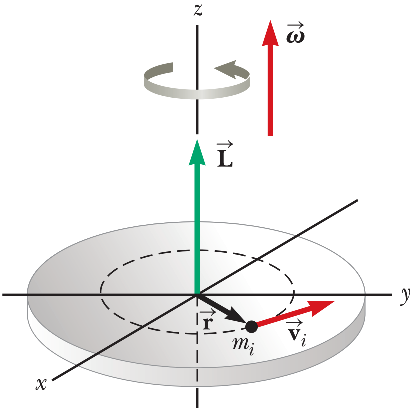
Angular Momentum of Rigid Object



All parts of the object have the same angular velocity $\vec{\omega}$.

Angular Momentum of Rigid Object

Consider a rotating rigid object. Let the z-axis point along the axis of rotation, $\vec{\omega} = \omega\hat{\mathbf{k}}$.

For one particle in the object, mass m_i, at radius r_i:

$$\begin{aligned} \vec{\mathbf{L}} &= m_i v_i r_i \,\hat{\mathbf{k}} \\ &= m_i(\omega r_i) r_i \,\hat{\mathbf{k}} \\ &= m_i r_i^2 \vec{\omega} \end{aligned}$$

For a rigid object made of many particles:

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\end{aligned}$$

For a rigid object made of many particles:

$$\begin{aligned}
\vec{\mathbf{L}}_{\text{tot}} &= \sum_i \vec{\mathbf{L}}_i \\
&= \left(\sum_i m_i r_i^2\right)\vec{\omega} \\
&= I\vec{\omega}
\end{aligned}$$

Angular Momentum of Rigid Object

For a rigid object:

$$\vec{\mathsf{L}} = I\vec{\omega}$$

where I is the moment of inertia and $\vec{\omega}$ is the angular velocity.

Question

Quick Quiz 11.3[1] A solid sphere and a hollow sphere have the same mass and radius. They are rotating with the same angular speed. Which one has the higher angular momentum?

(A) the solid sphere

(B) the hollow sphere

(C) both have the same angular momentum

(D) impossible to determine

[1] Serway & Jewett, page 343.

Question

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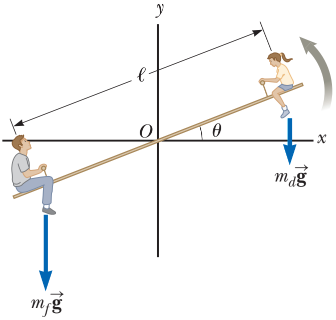
Example 11.6 - Rigid Object Angular Momentum

A father of mass m_f and his daughter of mass m_d sit on opposite ends of a seesaw at equal distances from the pivot at the center. The seesaw is modeled as a rigid rod of mass M and length ℓ, and is pivoted without friction. At a given moment, the combination rotates in a vertical plane with an angular speed ω.

Find an expression for the magnitude of the system's angular momentum.



Example 11.6 - Rigid Object Angular Momentum

Magnitude of the system's angular momentum?

Use: $L = I\omega$

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Magnitude of the system's angular momentum?

Use: $L = I\omega$

$$I = \frac{1}{12}M\ell^2 + m_f\left(\frac{\ell}{2}\right)^2 + m_d\left(\frac{\ell}{2}\right)^2$$

So,

$$L = \frac{\ell^2}{4}\left(\frac{1}{3}M + m_f + m_d\right)\omega$$

Angular Momentum and Newton's Second Law

Once again:

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Confirming $\frac{d\vec{L}}{dt} = \tau_{net}$:

Angular Momentum and Newton's Second Law

Confirming $\frac{d\vec{\mathbf{L}}}{dt} = \tau_{net}$:

$$\frac{d\vec{\mathbf{L}}}{dt} = \frac{d(\vec{\mathbf{r}} \times \vec{\mathbf{p}})}{dt}$$

Angular Momentum and Newton's Second Law

Confirming $\frac{d\vec{\mathbf{L}}}{dt} = \tau_{net}$:

$$\frac{d\vec{\mathbf{L}}}{dt} = \frac{d(\vec{\mathbf{r}} \times \vec{\mathbf{p}})}{dt}$$
$$= \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} + \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{p}}$$
$$= \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} + \cancelto{0}{\vec{\mathbf{v}} \times \vec{\mathbf{p}}}$$

because, $\vec{\mathbf{v}}$ and $\vec{\mathbf{p}}$ are always parallel. Noting $\vec{\mathbf{F}}_{net} = \frac{d\vec{\mathbf{p}}}{dt}$:

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{net}$$

Angular Momentum and Newton's Second Law

Confirming $\frac{d\vec{\mathbf{L}}}{dt} = \tau_{net}$:

$$\begin{aligned} \frac{d\vec{\mathbf{L}}}{dt} &= \frac{d(\vec{\mathbf{r}} \times \vec{\mathbf{p}})}{dt} \\ &= \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} + \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{p}} \\ &= \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} + \cancelto{0}{\vec{\mathbf{v}} \times \vec{\mathbf{p}}} \end{aligned}$$

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$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{net}$$

By definition $\vec{\boldsymbol{\tau}}_{net} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{net}$, so

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\boldsymbol{\tau}}_{net} \qquad \checkmark$$

Angular Impulse

We have:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{dt}}$$

We can now define the *angular impulse* on a system as

$$\overrightarrow{\Delta \boldsymbol{L}} = \int \vec{\boldsymbol{\tau}}_{\text{net}}\,\mathrm{dt}$$

A torque applied over time changes the angular momentum.

- Angular impulse is a vector
- Units: kg m^2 s^{-1}

Newton's Second Law for Rotations

We said

$$\vec{\tau}_{\text{net}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t}$$

is the more general form of Newton's second law for rotations.

What about our previous version of Newton's second law for rotations ($\vec{\tau}_{\text{net}} = I\,\vec{\alpha}$)? Suppose I is constant:

$$\vec{\tau}_{\text{net}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \frac{\mathrm{d}(I\vec{\omega})}{\mathrm{d}t}$$

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$$\begin{aligned}
\vec{\boldsymbol{\tau}}_{\text{net}} &= \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{dt}} = \frac{\mathrm{d}(I\vec{\boldsymbol{\omega}})}{\mathrm{dt}} \\
&= \cancelto{0}{\frac{\mathrm{d}I}{\mathrm{dt}}}\vec{\boldsymbol{\omega}} + I\frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{dt}
\end{aligned}$$

If we consider a fixed rigid object and fixed axis of rotation, $\frac{\mathrm{d}I}{\mathrm{dt}} = 0$:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = I\,\vec{\boldsymbol{\alpha}}$$

Summary

- introducing angular momentum
- angular momentum of rigid objects
- Newton's second law again
- angular impulse

Assignment 4 due Friday, Mar 20.

(Uncollected) Homework Serway & Jewett:

- Ch 11, onward from page 355. Probs: 1, 3, 11, 13, 17, 25