Rotation
Angular Momentum
Conservation of Angular Momentum

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Last time

- Definition of angular momentum
- angular momentum of rigid objects
- relation to Newton's 2nd law

Overview

- angular impulse
- angular momentum of an object moving in a straight line
- conservation of angular momentum
- applying conservation of momentum

Angular Momentum and Newton's Second Law

More general form of Newton's second law for rotations:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \frac{d\vec{\mathbf{L}}}{dt}$$

$\Rightarrow$ torques cause changes in angular momentum, $\vec{\mathbf{L}}$.

Compare: $\vec{\mathbf{F}}_{\text{net}} = \frac{d\vec{\mathbf{p}}}{dt}$

Angular Impulse

We have:

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t}$$

We can now define the *angular impulse* on a system as

$$\overrightarrow{\Delta \boldsymbol{L}} = \int \vec{\boldsymbol{\tau}}_{\text{net}}\, \mathrm{d}t$$

A torque applied over time changes the angular momentum.

- Angular impulse is a vector
- Units: kg m^2 s^{-1}

Newton's Second Law for Rotations

We said

$$\vec{\boldsymbol{\tau}}_{\text{net}} = \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t}$$

is the more general form of Newton's second law for rotations.

What about our previous version of Newton's second law for rotations ($\vec{\boldsymbol{\tau}}_{\text{net}} = I\,\vec{\boldsymbol{\alpha}}$)? Suppose I is constant:

$$\vec{\boldsymbol{\tau}}_{\text{net}} \;=\; \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \frac{\mathrm{d}(I\vec{\boldsymbol{\omega}})}{\mathrm{d}t}$$

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$$\begin{aligned} \vec{\tau}_{\text{net}} &= \frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{dt}} = \frac{\mathrm{d}(I\vec{\omega})}{\mathrm{dt}} \\ &= \cancelto{0}{\frac{\mathrm{d}I}{\mathrm{dt}}}\vec{\omega} + I\frac{\mathrm{d}\vec{\omega}}{dt} \end{aligned}$$

If we consider a fixed rigid object and fixed axis of rotation, $\frac{\mathrm{d}I}{\mathrm{dt}} = 0$:

$$\vec{\tau}_{\text{net}} = I\,\vec{\alpha}$$

Angular Momentum of a Falling Object

A rock, mass m, is dropped near the surface of the Earth at the north pole from a height h. After a time t its velocity is v. Let R_{Earth} be the radius of the Earth.

What is the magnitude of angular momentum of the rock about the center of the Earth?

(A) mgh

(B) mvh

(C) mvR_{Earth}

(D) 0

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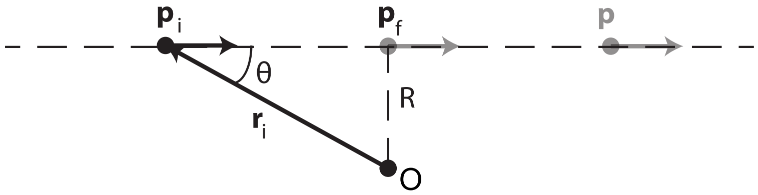
Angular mtm of an object moving in a Straight Line

Suppose a particle is in motion relative to an axis O and no forces act on it. ($\Delta \vec{\mathbf{p}} = 0$)

The distance of closest approach of the particle to the axis is R.



What is the angular momentum of the particle about O?

Isolated Object moving in a Straight Line

First consider the particle at point f. $\vec{\mathbf{p}}$ is perpendicular to $\vec{\mathbf{r}}_f$.

$$\begin{aligned}\vec{\mathbf{L}}_f &= \vec{\mathbf{r}}_f \times \vec{\mathbf{p}} \\ &= (R)(mv)\sin(90^\circ)\,(-\hat{\mathbf{k}}) \\ &= mvR\,(-\hat{\mathbf{k}})\end{aligned}$$

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Does this agree with what $\vec{\mathbf{L}}$ is at point i?

$$\vec{\mathbf{L}}_i = \vec{\mathbf{r}}_i \times \vec{\mathbf{p}} = (r_i)(mv)\sin\theta_i\,(-\hat{\mathbf{k}})$$

But, $r_i \sin\theta_i = R$, so

$$\vec{\mathbf{L}}_i = mvR\,(-\hat{\mathbf{k}})$$

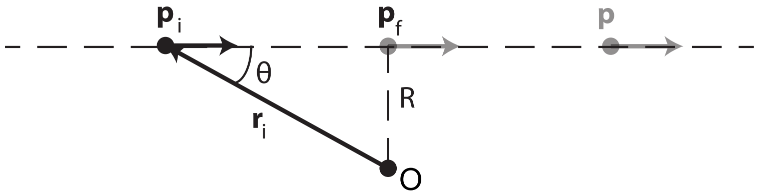
so yes, $\vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f$.

Angular mtm of an object moving in a Straight Line

The magnitude of the angular momentum of an object, of mass m and velocity v, traveling in a straight line about an axis O is

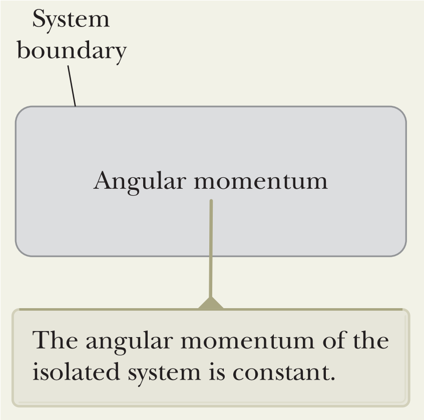
$$L = mvR$$

where R is the **distance of closest approach** of the axis O.



Conservation of Angular Momentum

For an *isolated* system, *ie.* a system with no external torques, total angular momentum is *conserved*.



[1]Figures from Serway & Jewett.

Conservation of Angular Momentum

For zero net external torque:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\sum_i \vec{\mathbf{L}}_i\right) = 0$$

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This corresponds to a rotational symmetry in the equations of motion.

(And now "isolated" means no external torques act on the system.)

Conservation of Angular Momentum

Suppose we have a collection of massive particles, which together have some moment of inertia I_i at time $t = t_i$, and an angular velocity $\vec{\omega}_i$.

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Then, even if the arrangement of the particles is changing $(I(t))$:

$$I_i \vec{\omega}_i = I_f \vec{\omega}_f$$

Example 11.7 - Conservation of Angular Momentum

A star rotates with a period of 30 days about an axis through its center. The period is the time interval required for a point on the star's equator to make one complete revolution around the axis of rotation. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4 km, collapses into a neutron star of radius 3.0 km.

Determine the period of rotation of the neutron star.

[0] Serway & Jewett, page 347.

Example 11.7 - Conservation of Angular Momentum

Period of rotation of neutron star?

Use: $I_i \vec{\omega}_i = I_f \vec{\omega}_f$ and $\omega = \frac{2\pi}{T}$

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$$I_i = \frac{2}{5} M \left(1.0 \times 10^7 \text{ m}\right)^2 \; ; \quad I_f = \frac{2}{5} M \left(3.0 \times 10^3 \text{ m}\right)^2$$

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$$\begin{aligned} I_i \, \frac{2\pi}{T_i} &= I_f \, \frac{2\pi}{T_f} \\ T_f &= \frac{I_f}{I_i} T_i \\ &= \frac{(3.0 \times 10^3 \text{ m})^2}{(1.0 \times 10^7 \text{ m})^2} \,(30 \times 24 \times 3600 \text{ s}) \\ &= 0.23 \text{ s} \end{aligned}$$

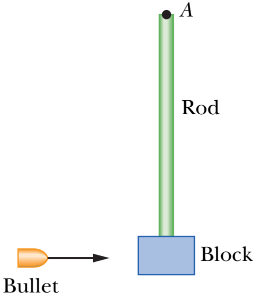
Another Example

A bullet of mass m_b is fired into a block of mass M attached to the end of a uniform rod of mass m_r and length ℓ. The block-rod-bullet system then rotates in the plane of the figure, about a fixed axis at A. Treat the block as a particle.

(a) What is the rotational inertia of the block-rod-bullet system, when the bullet is lodged in the block, about point A?

(b) If the angular speed of the system about A just after impact is ω, what is the bullet's speed just before impact?

(c) What if we try to conserve linear momentum?



[1]Halliday, Resnick, Walker, 9th ed, page 302, variation on #60.

Another Example

Let $r = 0.6$ m be the length of the rod.

(a) rotational inertia about A with bullet embedded?

$$\begin{aligned} I_A &= I_{\text{bullet}} + I_{\text{block}} + I_{\text{rod}} \\ &= m_b \ell^2 + M\ell^2 + \frac{1}{3} m_r \ell^2 \\ &= (m_b + M + \frac{m_r}{3})\ell^2 \end{aligned}$$

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(b) bullet's initial speed, v_i?

isolated system $\Rightarrow$ conserve angular momentum in collision

$$\begin{aligned} L_{A,i} &= L_{A,f} \\ L_{\text{bullet},A,i} + L_{\text{block+rod},A,i} &= I_{A,f}\omega_f \\ m_b v_i \ell \quad + \quad 0 &= I_A\omega \\ v_i &= \frac{(m_b + M + \frac{m_r}{3})\ell\omega}{m_b} \end{aligned}$$

Summary

- angular impulse
- angular momentum of an object moving in a straight line
- conservation of angular momentum

Final Exam Tuesday, Mar 24, via Canvas & Zoom, be ready at 9am.

4th Assignment due this Friday.

(Uncollected) Homework

- Play with office furniture

Serway & Jewett:

- Ch 11, onward from page 357. Probs: 31, 33, 37, 39, 41