Oscillations
Simple Harmonic Motion

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Mar 18, 2020

Last Time

- torque and changes in angular momentum

Overview

- oscillations
- simple harmonic motion (SHM)
- spring systems
- energy in SHM
- pendula - did not get to

Oscillations and Periodic Motion

Many physical systems exhibit cycles of repetitive behavior.

After some time, they return to their initial configuration.

Examples:

- clocks
- rolling wheels
- a pendulum
- bobs on springs

Oscillations

oscillation

motion that repeats over a period of time

amplitude

the magnitude of the vibration; how far does the object move from its average (equilibrium) position.

period

the time for one complete oscillation.

After 1 period, the motion repeats itself.

Simple Harmonic Motion

The oscillations of bobs on springs and pendula are very regular and simple to describe.

It is called simple harmonic motion.

simple harmonic motion (SHM)

any motion in which the acceleration is proportional to the displacement from equilibrium, but opposite in direction

The force causing the acceleration is called the “restoring force”.

SHM and Springs

If a mass is attached to a spring, the force on the mass depends on its displacement from the spring's natural length.

Hooke's Law:

$$\vec{\mathbf{F}} = -k\,\vec{\mathbf{x}}$$

where k is the spring constant and x is the displacement (position) of the mass.

If the spring is compressed and then the mass is released, it will move outward until is stretches the spring by the same amount the spring was compressed initially before it bounces back again. (Assume no friction.)

Hooke's law gives the force on the bob $\Rightarrow$ SHM.

The spring force is the *restoring force*.

SHM and Springs

How can we find an equation of motion for the block?

Newton's second law:

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_s = m\vec{\mathbf{a}}$$

Using the definition of acceleration: $a_x = \frac{\mathrm{d}^2 x}{\mathrm{dt}^2}$

$$\frac{\mathrm{d}^2 x}{\mathrm{dt}^2} = -\frac{k}{m}x$$

Define

$$\omega = \sqrt{\frac{k}{m}}$$

and we can write this equation as:

$$\frac{\mathrm{d}^2 x}{\mathrm{dt}^2} = -\omega^2 x$$

SHM and Springs

To solve:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x$$

notice that it is a second order linear differential equation.

We can actually find the solutions just be inspection.

A solution $x(t)$ to this equation has the property that if we take its derivative twice, we get the same form of the function back again, but with an additional factor of $-\omega^2$.

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$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 \left(A \ \cos(\omega t)\right) \qquad \checkmark$$

SHM and Springs

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

In fact, any solutions of the form:

$$x = B_1 \cos(\omega t + \phi_1) + B_2 \sin(\omega t + \phi_2)$$

where B_1, B_2, ϕ_1, and ϕ_2 are constants.

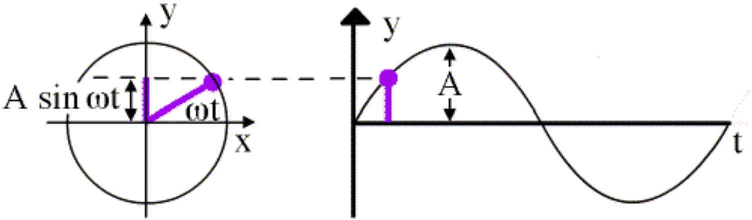
However, since $\sin(\theta) = \cos(\theta + \pi/2)$, in general any solution can be written in the form:

$$x = A\cos(\omega t + \phi)$$

Oscillating Solutions

$$x = A\cos(\omega t)$$

(or equivalently, $y = A\sin(\omega t)$.)



We now call ω the **angular frequency** of the oscillation.

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[1]Figure from School of Physics webpage, University of New South Wales.

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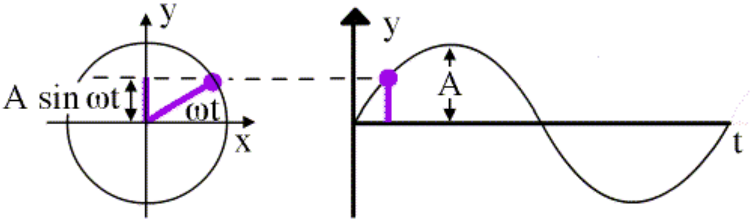

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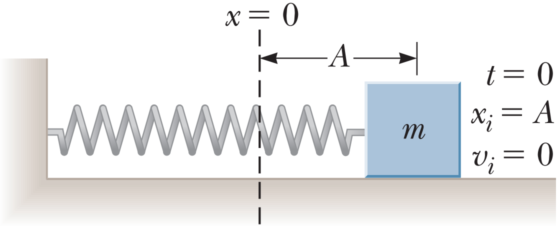
SHM and Springs

The position of the bob at a given time is given by:

$$x = A\cos(\omega t + \phi)$$

A is the amplitude of the oscillation. We could also write $x_{\text{max}} = A$.



The speed of the particle at any point in time is:

$$v = -A\omega\sin(\omega t + \phi)$$

We find that by differentiating the expression for position.

SHM and Springs

$$x = A\cos(\omega t + \phi)$$

ω is the angular frequency of the oscillation.

When $t = \frac{2\pi}{\omega}$ the block has returned to the position it had at $t = 0$. That is one complete cycle.

SHM and Springs

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When $t = \frac{2\pi}{\omega}$ the block has returned to the position it had at $t = 0$. That is one complete cycle.

Recalling that $\omega = \sqrt{k/m}$:

$$\text{Period, } T = 2\pi\sqrt{\frac{m}{k}}$$

Only depends on the mass of the bob and the spring constant.
Does not depend on the amplitude.

SHM and Springs Question

A mass-spring system has a period, T. If the mass of the bob is quadrupled (and everything else is unchanged), what happens to the period of the motion?

(A) halves, $T/2$

(B) remains unchanged, T

(C) doubles, $2T$

(D) quadruples, $4T$

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Oscillations and Waveforms

Any oscillation can be plotted against time. *eg.* the position of a vibrating object against time.

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From this wave description of the motion, a lot of parameters can be specified.

This allows us to quantitatively compare one oscillation to another.

Examples of quantities: period, amplitude, frequency.

Measuring Oscillations

frequency

The number of complete oscillations in some amount of time. Usually, oscillations per second.

$$f = \frac{1}{T}$$

Units of frequency: Hertz. $1\ \mathrm{Hz} = 1\ \mathrm{s}^{-1}$

If one oscillation takes a quarter of a second (0.25 s), then there are 4 oscillations per second. The frequency is $4\ \mathrm{s}^{-1} = 4\ \mathrm{Hz}$.

$$\omega = 2\pi f$$

Period and frequency question

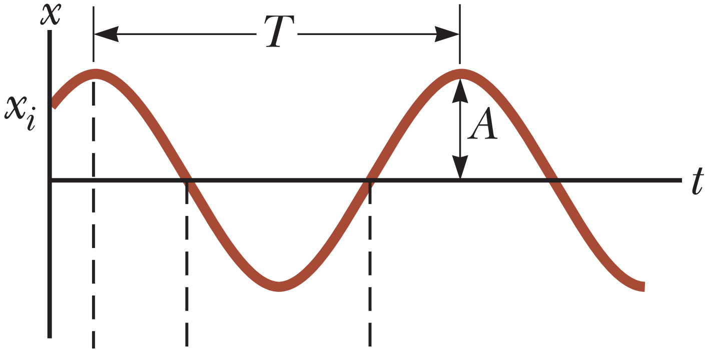
What is the period, in seconds, that corresponds to each of the following frequencies?

1. 10 Hz
2. 0.2 Hz
3. 60 Hz

[1] Hewitt, page 350, Ch 18, problem 2.

Waveform

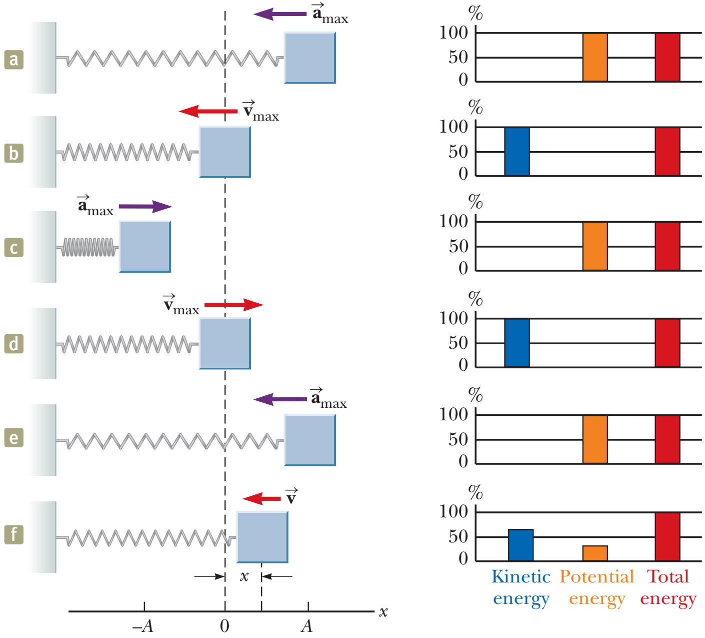
$$x = A\cos(\omega t + \phi)$$



$$f = \frac{1}{T}$$

[1]Figure from Serway & Jewett, 9th ed.

Energy in SHM



Energy in SHM

Potential Energy:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

Kinetic Energy:

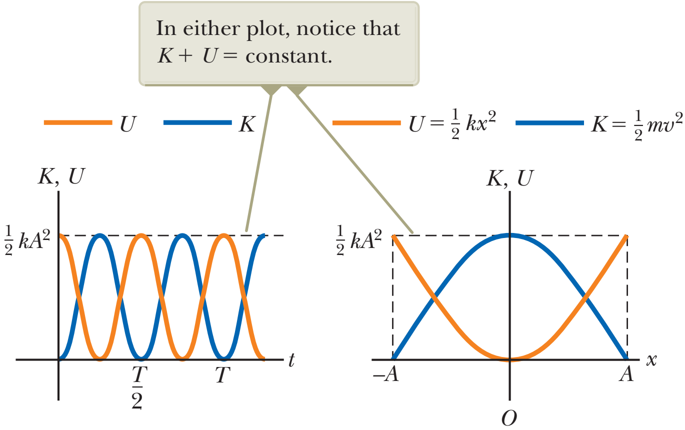
$$K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$$

Using $\omega^2 = k/m$

$$\begin{aligned} K + U &= \frac{1}{2}kA^2\big(\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\big) \\ &= \frac{1}{2}kA^2 \end{aligned}$$

This does not depend on time!

Energy in SHM



$$K + U = \frac{1}{2}kA^2$$

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Pendula and SHM - Did not get to

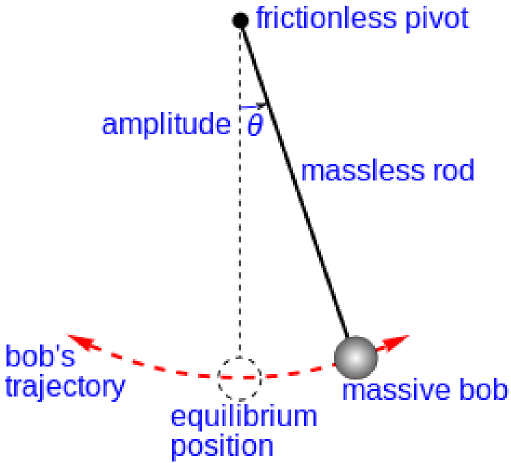
pendulum

a massive bob attached to the end rod or string that will oscillate along a circular arc under the influence of gravity

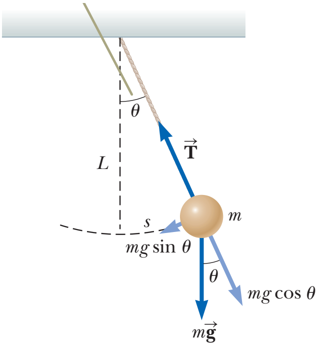
A pendulum bob that is displaced to one side by a small amount and released follows SHM to a good approximation.

Gravity and the tension in the string provide the *restoring force*.

Pendula and SHM - Did not get to



Pendula and SHM



The tangential component of the net force on the pendulum bob is $-mg \sin \theta$ along the arc s, when the string is at an angle θ to the vertical.

Therefore, Newton's second law for rotations ($\tau_{net} = I\alpha$) analyzing about the pivot gives:

$$I \frac{d^2\theta}{dt^2} = -Lmg \sin \theta$$

Pendula and SHM

For a "simple pendulum" $I = mL^2$ (string is massless).

$$mL^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mgL \sin\theta$$

Dividing by mL^2:

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This is a non-linear second order differential equation, and the exact solution is a bit ugly.

However, we can approximate this motion as SHM. (Remember, for SHM acceleration is proportional to displacement but in the opposite direction.)

Pendula and SHM

For small values of θ (measured in radians!), $\sin\theta \approx \theta$.

That means for small oscillations of a pendulum, the restoring force on the bob is roughly:

$$F = -mg\theta$$

and the motion is SHM.

Our equation of motion becomes:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2\theta$$

where $\omega = \sqrt{\frac{g}{L}}$

Pendula and SHM

Equation of motion:

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

We already know the solutions should be of the form

$$\theta = \theta_{max} \cos(\omega t + \phi)$$

where θ_{max} is the amplitude and $T = \frac{2\pi}{\omega}$ is the period.

Pendula and SHM

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$$\theta = \theta_{\max}\cos(\omega t + \phi)$$

where $\theta_{\max}$ is the amplitude and $T = \frac{2\pi}{\omega}$ is the period.

Recalling that $\omega = \sqrt{g/L}$, for the simple pendulum:

Period, $T = 2\pi\sqrt{\frac{L}{g}}$

Problem

An astronaut on the Moon attaches a small brass ball to a 1.00 m length of string and makes a simple pendulum. She times 15 complete swings in a time of 75 seconds. From this measurement she calculates the acceleration due to gravity on the Moon. What is her result?[1]

[1] Hewitt, "Conceptual Physics", problem 8, page 350.

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1.58 m/s^2

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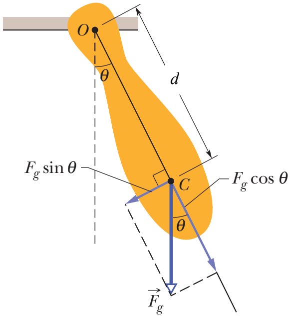
"Physical Pendulum"
Skipped, Not on Final



The swinging rigid object has a moment of inertia I about the pivot.

As before, for small angles Newton's second law for rotations ($\tau_{net} = I\alpha$) analyzing about the pivot gives:

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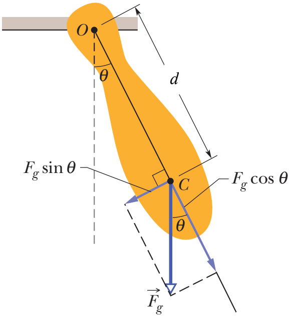

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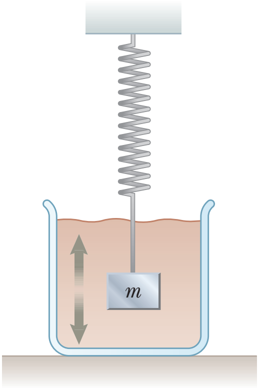
Solving yields $\theta = \theta_{max} \cos(\omega t + \phi)$, with

$$\omega = \sqrt{mgd/I} \; ; \quad T = 2\pi \sqrt{\frac{I}{mgd}}$$

Damped Oscillations - Skipped! Not on Final

Imagine now a system with a restoring force, but also a resistive force.



In the picture, that would be a fluid resistance force, $\vec{\mathbf{R}} = -b\vec{\mathbf{v}}$.

$$F_{\text{net}} = -kx - b\,\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

Damped Oscillations - Skipped! Not on Final

$$F_{\text{net}} = -kx - b\,\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{b}{m}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = 0$$

Solutions[1] are

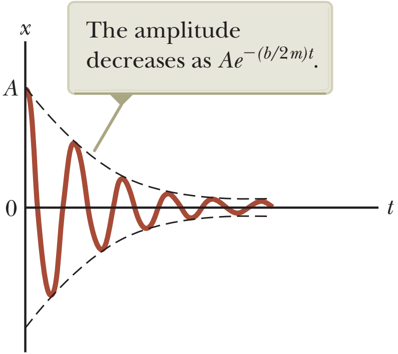
$$x = A\,e^{-(b/2m)t}\cos(\omega t + \phi)$$

where $\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$ and A and ϕ are constants.

[1] See the appendix to these slides for proof.

Damped Oscillations -Skipped! Not on Final

$$x = A\, e^{-(b/2m)t} \cos(\omega t + \phi)$$



The decaying exponential $e^{-(b/2m)t}$ shows that the amplitude of the waveform is decreasing (energy lost). The cosine is an oscillating solution.

Summary

- oscillations
- simple harmonic motion (SHM)
- spring and pendulum systems

4th Assignment due Friday, Mar 20.

Canvas Quiz/Survey due Thursday night, not posted yet, will take ~5 mins, get credit for it as a quiz!

Final Exam Tuesday, Mar 24, via Canvas & Zoom, be ready at 9am.

(Uncollected) Homework

Serway & Jewett,

- Ch 15, onward from page 472. OQs: 13; CQs: 5, 7; Probs: 1, 3, 9, 35, 41

Appendix: Damped Oscillations Solution Derivation

$$\frac{\mathrm{d}^2 x}{\mathrm{dt}^2} + \frac{b}{m}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{k}{m}x = 0$$

Suppose an exponential function is the solution to this equation:

$$x = B\,e^{rt}$$

r and B are constants.
Then

$$B\,e^{rt}(r^2 + \frac{b}{m}r + \frac{k}{m}r) = 0$$

The exponential function is not zero for any finite t, so the other factor must be zero. We must find the roots for r that make this equation true.

Damped Oscillations Solution Derivation

This is called the characteristic equation

$$r^2 + \frac{b}{m} r + \frac{k}{m} r = 0$$

The roots are:

$$r = \frac{-b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

This means the solutions are of the form:

$$x = e^{-b/(2m)t} \left(B_1 e^{i\omega t} + B_2 e^{-i\omega t}\right)$$

where

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

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Recall that $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$. (If you haven't seen this, try to prove it using the series expansions of cosine and the exponential function.)

We can write the solution as

$$x = A\, e^{-(b/2m)t} \cos(\omega t + \phi)$$

where $B_1 = A\, e^{i\phi}$ and $B_2 = A\, e^{-i\phi}$.

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