Electricity and Magnetism
Mutual Inductance
Oscillations in Circuits

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Last time

- energy stored in an inductor
- coaxial inductor
- mutual inductance
Overview

- mutual inductance applications
- LC circuits
- RLC circuits
**Mutual Inductance**

An inductor can have an induced emf from its own changing magnetic field.

It also can have an emf from an external changing field.

That external changing field could be another inductor.
Mutual Inductance

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For self-inductance on a coil labeled 1:

\[ N_1 \Phi_{B,1} = L_1 i_1 \]

For mutual inductance:

\[ N_1 \Phi_{B,2\rightarrow1} = M_{21} i_2 \]

The flux is in coil 1, but the current that causes the flux is in coil 2.
Mutual Inductance

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\[ M = \frac{N_1 \Phi_{B,2 \rightarrow 1}}{i_2} = \frac{N_2 \Phi_{B,1 \rightarrow 2}}{i_1} \]

which has the same form as Eq. 30-28, the definition of inductance. We can recast Eq. 30-57 as

\[ M_{21} i_1 = \frac{N_1}{H_{11005}} \frac{N_2}{H_{9021}} i_2 \]

(30-59)

If we cause \( i_1 \) to vary with time by varying \( R \), we have

\[ (30-60) \]

The right side of this equation is, according to Faraday’s law, just the magnitude of the emf appearing in coil 2 due to the changing current in coil 1. Thus, with a minus sign to indicate direction,

\[ (30-61) \]

which you should compare with Eq. 30-35 for self-induction (814

(30-62)

Let us now interchange the roles of coils 1 and 2, as in Fig. 30-19b; that is, we set up a current \( i_2 \) in coil 2 by means of a battery, and this produces a magnetic flux \( \Phi_{B,1 \rightarrow 2} \) that links coil 1. If we change \( i_2 \) with time by varying \( R \), we then have, by the argument given above,

\[ (30-62) \]

Thus, we see that the emf induced in either coil is proportional to the rate of change of current in the other coil. The proportionality constants \( M_{21} \) and \( M_{12} \) seem to be different. We assert, without proof, that they are in fact the same so that no subscripts are needed. (This conclusion is true but is in no way obvious.) Thus, we have

\[ M_{21} = M_{12} \]

and we can rewrite Eqs. 30-61 and 30-62 as

\[ (30-64) \]

and

\[ (30-65) \]

\[ \frac{N_1}{H_{11005}} \frac{N_2}{H_{9021}} M \frac{di_1}{dt} \]

\[ \frac{N_1}{H_{11005}} \frac{N_2}{H_{9021}} M \frac{di_2}{dt} \]
Mutual Inductance

\[ N_1 \Phi_{B,2 \rightarrow 1} = M_{21} i_2 \]

Considering the rate of change of both sides with time, and using Faraday’s Law \( \varepsilon = -\frac{d\Phi_B}{dt} \),

\[ \varepsilon_1 = -M \frac{di_2}{dt} \]

and

\[ \varepsilon_2 = -M \frac{di_1}{dt} \]

A change of current in one coil causes a magnetic flux in the other.
Mutual Inductance

Imagine the two coils are moved closer together, with the orientation of both coils remaining fixed.

Because of this movement, the mutual induction of the two coils

(A) increases
(B) decreases
(C) is unaffected
Mutual Inductance

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(A) increases
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Mutual Inductance Applications

If there is a changing current in one coil, an emf can be induced in the other coil.

The current can be transferred to a whole different circuit that is no directly connected.
Mutual Inductance Applications

If there is a changing current in one coil, an emf can be induced in the other coil.

The current can be transferred to a whole different circuit that is no directly connected.

This can be used for **wireless charging**.

It is also used in **transformers**: devices that change the voltage and current of a power supply.

Other applications include **sensors**, particularly traffic light sensors and pulse induction metal detectors.

For any of these applications to work, there must be a constantly changing current.
Mutual Inductance Applications: Wireless Charger (Ex 32.5)

Electric toothbrush: Model the base of the charger as a solenoid of length $\ell$, with $N_B$ turns, carrying a current $i$, and having a cross-sectional area $A$. The handle coil contains $N_H$ turns and completely surrounds the base coil. Find the mutual inductance of the system.
Mutual Inductance Applications: Wireless Charger (Ex 32.5)

\[ M = \frac{N_H \Phi_{B,H}}{i_B} \]

Must find an expression for \( \Phi_{B,H} \).
Mutual Inductance Applications: Wireless Charger (Ex 32.5)

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Must find an expression for \( \Phi_{B,H} \).

\[ \Phi_{B,H} = \Phi_{B,B} \]

\[ = B \cdot A \]

\[ = \mu_0 i_b \frac{N_B}{\ell} A \]
Mutual Inductance Applications: Wireless Charger (Ex 32.5)

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Must find an expression for \( \Phi_{B,H} \).

\[ \Phi_{B,H} = \Phi_{B,B} \]
\[ = \mathbf{B} \cdot \mathbf{A} \]
\[ = \mu_0 i_b \frac{N_B}{\ell} A \]

So,

\[ M = \frac{\mu_0 N_H N_B A}{\ell} \]
Interesting time-dependent behavior of charge and current also occurs in circuits with inductors and capacitors.

The capacitor is first charged, then put into a circuit with the inductor.
Assume the resistance of the wires is zero, then electromagnetic energy is conserved in the circuit.

The energy is stored either in the E-field in the capacitor or the B-field in the inductor.

\[ U_{\text{tot}} = \frac{q^2}{2C} + \frac{Li^2}{2} \]
Energy is conserved $\Rightarrow \quad \frac{dU_{\text{tot}}}{dt} = 0$.

$$q \frac{dq}{C \ dt} + Li \frac{di}{dt} = 0$$

Remembering that $i = \frac{dq}{dt}$:

$$\frac{q \ dq}{C \ dt} + L \frac{dq}{dt} \frac{d^2q}{dt^2} = 0$$
LC Circuits and Oscillations

Energy is conserved ⇒ $\frac{dU_{\text{tot}}}{dt} = 0$.

\[ \frac{q}{C} \frac{dq}{dt} + Li \frac{di}{dt} = 0 \]

Remembering that $i = \frac{dq}{dt}$:

\[ \frac{q}{C} \frac{dq}{dt} + L \frac{dq}{dt} \frac{d^2q}{dt^2} = 0 \]

This is a second order differential equation in $q$.

\[ \frac{d^2q}{dt^2} = -\frac{1}{LC} q \]

This is the equation for simple harmonic motion.
The solutions are oscillations in time:

\[ q(t) = Q_{\text{max}} \cos(\omega t + \phi) \]

where

\[ \omega = \frac{1}{\sqrt{LC}} \]
The charge $q$ and the current $i$ are 90° out of phase with each other.

$Q_{\text{max}}$ $I_{\text{max}}$

Figure 32.12 Graphs of charge versus time and current versus time for a resistanceless, nonradiating LC circuit.

To determine the value of the phase angle $\theta$, let's examine the initial conditions, which in our situation require that at $t = 0$, $i = 0$, and $q = Q_{\text{max}}$. Setting $i = 0$ at $t = 0$ in Equation 32.23 gives $0 = 2vQ_{\text{max}}\sin \theta$, which shows that $\theta = 0$. This value for $\theta$ also is consistent with Equation 32.21 and the condition that $q = Q_{\text{max}}$ at $t = 0$. Therefore, in our case, the expressions for $q$ and $i$ are

$$q = Q_{\text{max}} \cos \omega t \quad (32.24)$$

$$i = -\omega Q_{\text{max}} \sin \omega t = -I_{\text{max}} \sin \omega t \quad (32.25)$$

Graphs of $q$ versus $t$ and $i$ versus $t$ are shown in Figure 32.12. The charge on the capacitor oscillates between the extreme values $Q_{\text{max}}$ and $-Q_{\text{max}}$, and the current oscillates between $I_{\text{max}}$ and $-I_{\text{max}}$. Furthermore, the current is 90° out of phase with the charge. That is, when the charge is a maximum, the current is zero, and when the charge is zero, the current has its maximum value.

Let's return to the energy discussion of the LC circuit. Substituting Equations 32.24 and 32.25 in Equation 32.18, we find that the total energy is

$$U = U_E + U_B = \frac{Q^2_{\text{max}}}{2C} - \frac{1}{2LI_{\text{max}}^2} \sin^2 \omega t \quad (32.26)$$

This expression contains all the features described qualitatively at the beginning of this section. It shows that the energy of the LC circuit continuously oscillates between energy stored in the capacitor's electric field and energy stored in the inductor's magnetic field. When the energy stored in the capacitor has its maximum value $\frac{Q^2_{\text{max}}}{2C}$, the energy stored in the inductor is zero. When the energy stored in the inductor has its maximum value $\frac{L I_{\text{max}}^2}{2}$, the energy stored in the capacitor is zero.

Plots of the time variations of $U_E$ and $U_B$ are shown in Figure 32.13. The sum $U_E + U_B$ is a constant and is equal to the total energy $Q_{\text{max}}^2 / 2C$, or $\frac{L I_{\text{max}}^2}{2}$. Analytical verification is straightforward. The amplitudes of the two graphs in Figure 32.13 must be equal because the maximum energy stored in the capacitor (when $I = 0$) must equal the maximum energy stored in the inductor (when $q = 0$). This equality is expressed mathematically as $Q_{\text{max}}^2 / 2C = LI_{\text{max}}^2$. The sum of the two curves is a constant and is equal to the total energy stored in the circuit.
Oscillations in an LC Circuit lead to the phenomenon of resonance. The same phenomenon is observed in the LC circuit. (See Section 33.7.)

A representation of the energy transfer in an LC circuit is shown in Figure 32.11. As mentioned, the behavior of the circuit is analogous to that of the particle in simple harmonic motion studied in Chapter 15. For example, consider the block–spring system shown in Figure 15.10. The oscillations of this system are shown at the right of Figure 32.11. The potential energy \( \frac{1}{2}kx^2 \) stored in the stretched spring is analogous to the potential energy \( \frac{Q_{\text{max}}^2}{2C} \) stored in the capacitor in Figure 32.11. The kinetic energy \( \frac{1}{2}mv^2 \) of the moving block is analogous to the magnetic energy \( \frac{1}{2}LI_{\text{max}}^2 \) stored in the inductor, which requires the presence of moving charges. In Figure 32.11a, all the energy is stored as electric potential energy in the capacitor at \( t = 0 \) (because \( i = 0 \)), just as all the energy in a block–spring system is initially stored as potential energy in the spring if it is stretched and released at \( t = 0 \). In Figure 32.11b, all the energy is stored as magnetic energy \( \frac{1}{2}LI_{\text{max}}^2 \) in the inductor, where \( I_{\text{max}} \) is the maximum current. Figures 32.11c and 32.11d show subsequent quarter-cycle situations in which the energy is all electric or all magnetic. At intermediate points, part of the energy is electric and part is magnetic.

\[ L \dot{i} + Q_{\text{max}} = 0 \]

\[ +Q_{\text{max}} \quad i = 0 \]

\[ -Q_{\text{max}} \quad C \]

\[ E \quad L \]

\[ k \]

\[ m \]

\[ v = 0 \]
Oscillations in an LC Circuit which leads to the phenomenon of resonance. The same phenomenon is observed in the LC circuit. (See Section 33.7.)

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$$E_{\text{inductor}} = \frac{1}{2} LI_{\text{max}}^2$$

$$E_{\text{capacitor}} = \frac{1}{2} C Q_{\text{max}}^2$$
Summary

- mutual inductance
- applications of mutual inductance
- LC circuits

Collected Homework 4! due Thursday, Mar 22.

Homework
Serway & Jewett:

- PREV: Ch 32, onward from page 988. Obj. Qs: 1; Conc. Qs.: 7; Probs: 11, 15, 19, 33, 41, 43
- NEW: Ch 32, Probs: 49