Electricity and Magnetism
AC Circuits
RLC Series Circuits
Impedance

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Last time

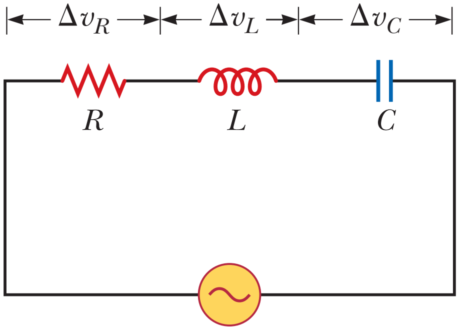
- phase offsets with inductors and capacitors
- reactance

Overview

- RLC series circuit
- impedance
- power
- resonance

AC in RLC Circuits

Consider the case of a *series* RLC circuit:



Suppose the emf source supplies:

$$\Delta v = \Delta V_{\text{max}} \sin(\omega t)$$

Can we relate this to the current that results?

$$i = I_{\text{max}} \sin(\omega t - \phi)$$

(Find ϕ and I_{max}.)

AC in RLC Circuits

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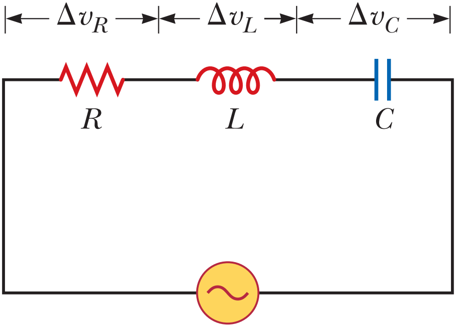


$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$$

AC in RLC Circuits

To follow along with the textbook's diagrams, we will now redefine our time variable, $t \to t'$.

Let

$$t' = t - \frac{\phi}{\omega}$$

so that $\omega t - \phi = \omega t'$.

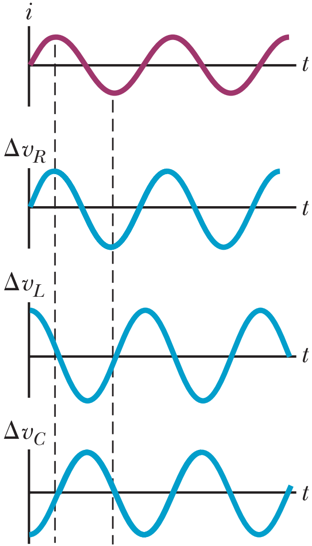
Then the voltage and current are given by:

$$\begin{aligned} \Delta v &= \Delta V_{\text{max}} \sin(\omega t' + \phi) \\ i &= I_{\text{max}} \sin(\omega t') \end{aligned}$$

AC in RLC Circuits

$$\Delta v = \Delta V_{\text{max}} \sin(\omega t' + \phi)$$
$$i = I_{\text{max}} \sin(\omega t')$$

$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$$



AC in RLC Circuits

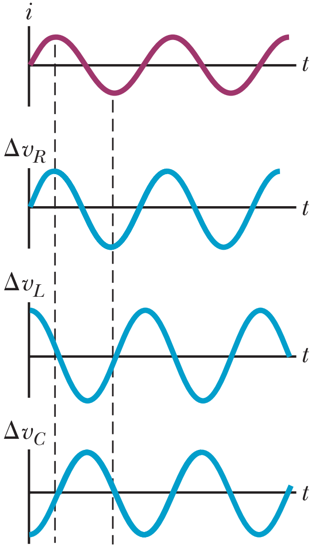
$$\Delta v = \Delta V_{\max} \sin(\omega t' + \phi)$$
$$i = I_{\max} \sin(\omega t')$$

$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$$

Δv_R is in phase with the current

$$\Delta v_R = \Delta V_R \sin(\omega t')$$

where $\Delta V_R = I_{\max} R$.



AC in RLC Circuits

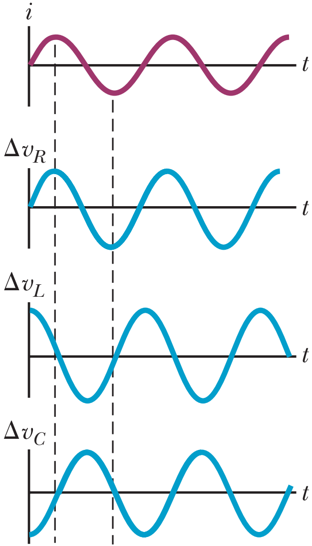
$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$$

Δv_L and Δv_C are π radians out of phase: easy to see how to add them:

$$\Delta v_L = \Delta V_L \sin(\omega t' + \pi/2)$$
$$\Delta v_C = \Delta V_C \sin(\omega t' - \pi/2)$$

Then,

$$\Delta v_L + \Delta v_C = (\Delta V_L - \Delta V_C) \cos \omega t'$$

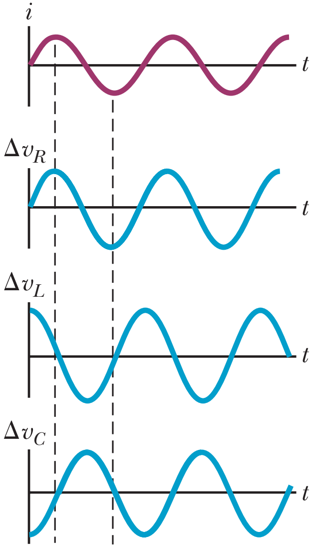


AC in RLC Circuits

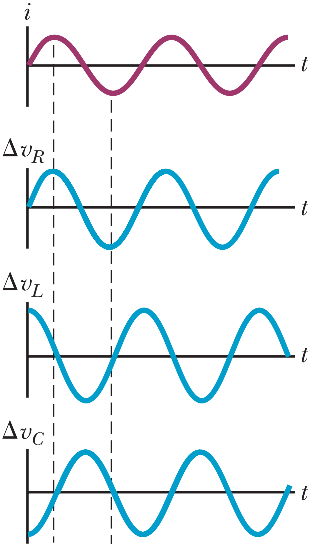
$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$$

However, $\Delta v_L + \Delta v_C$ and Δv_R are not in phase.

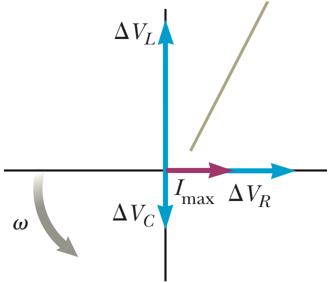
They can be added using a phasor diagram.



AC in RLC Circuits

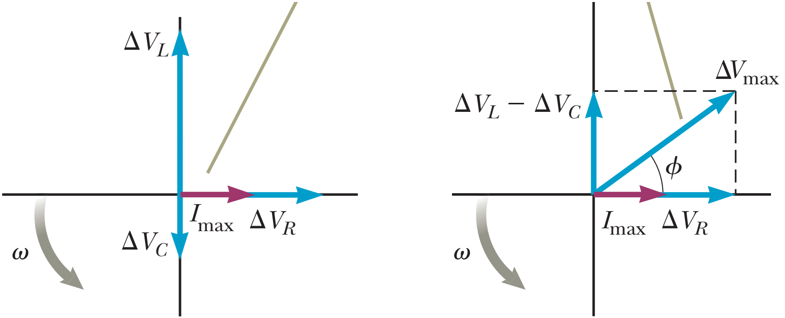


Phasor diagram:

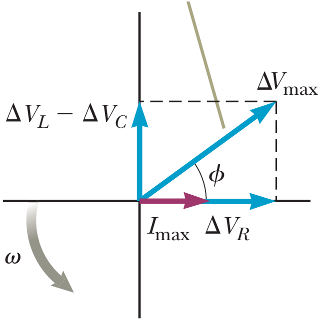


AC in RLC Circuits

Can add the vectors in the phasor diagram. (Normal vector addition.)



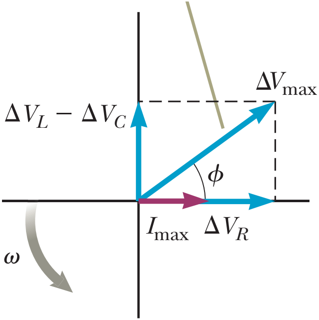
AC in RLC Circuits



Vector addition:

$$\Delta V_{max} = \sqrt{(\Delta V_R)^2 + (\Delta V_L - \Delta V_C)^2}$$

AC in RLC Circuits



Vector addition:

$$\Delta V_{max} = \sqrt{(\Delta V_R)^2 + (\Delta V_L - \Delta V_C)^2}$$

and

$$\Delta V_L - \Delta V_C = I_{max}(X_L - X_C)$$

So,

$$\Delta V_{max} = I_{max}\sqrt{R^2 + X^2}$$

where $X = X_L - X_C$

Notice that $\sqrt{R^2 + X^2}$ has units of Ohms and a role like a resistance...

Impedance

Impedance, Z

The ratio of the maximum voltage to the maximum current.

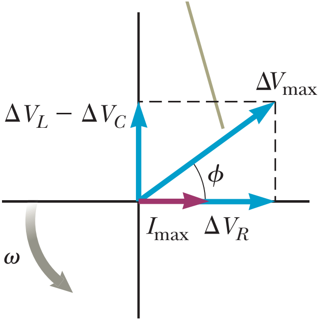
$$Z = \frac{\Delta V_{\text{max}}}{I_{\text{max}}}$$

For a series RLC circuit (or a component with some resistance, some capacitance, and some inductance):

$$Z = \sqrt{R^2 + X^2}$$

where the reactance $X = X_L - X_C$.

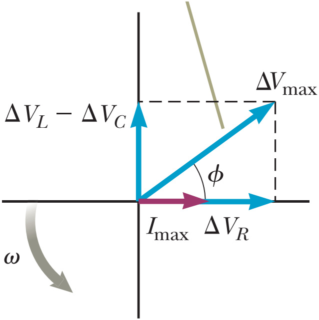
AC in RLC Circuits



Vector addition:

$$\tan \phi = (\Delta V_L - \Delta V_C)/(\Delta V_R)$$

AC in RLC Circuits



Vector addition:

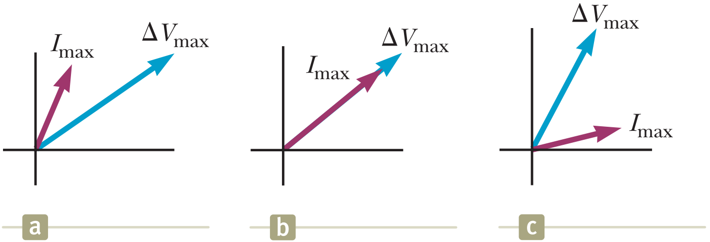
$$\tan \phi = (\Delta V_L - \Delta V_C)/(\Delta V_R)$$

$$\phi = \tan^{-1}\left(\frac{X}{R}\right)$$

(where the reactance $X = X_L - X_C$)

AC in RLC Circuits Question

For which of these is $X_C > X_L$?



(a) a

(b) b

(c) c

(d) none of these

AC in RLC Circuits Question

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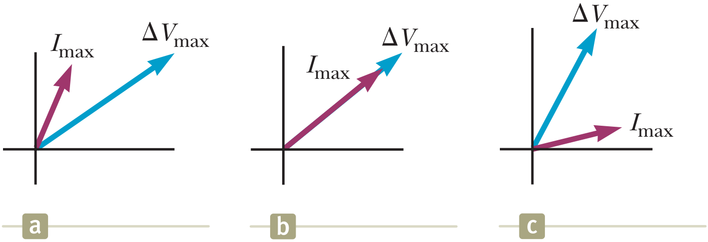

(a) a $\leftarrow$

(b) b

(c) c

(d) none of these

Power in AC Circuits

$$P = i\Delta v$$

$$\begin{aligned} \Delta v &= \Delta V_{\max} \sin(\omega t) \\ i &= I_{\max} \sin(\omega t - \phi) \end{aligned}$$

Therefore,

$$P = I_{\max} \Delta V_{\max} \sin(\omega t - \phi) \sin(\omega t)$$

Power in AC Circuits

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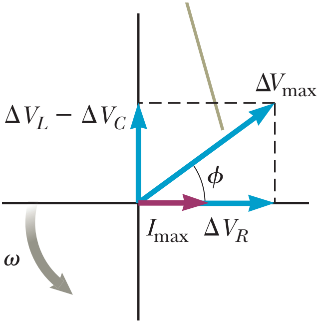
Therefore,

$$P = I_{\max} \Delta V_{\max} \sin(\omega t - \phi) \sin(\omega t)$$

After using some trigonometric identities:

$$P_{\text{avg}} = I_{\text{rms}} \Delta V_{\text{rms}} \cos\phi$$

AC in RLC Circuits



$$P_{avg} = I_{rms} \Delta V_{rms} \cos \phi$$

From the phasor diagram:

$$\Delta V_R = \Delta V_{max} \cos \phi$$

and since $\Delta V_R = I_{max} R$

$$\frac{I_{max} R}{\sqrt{2}} = \frac{\Delta V_{max}}{\sqrt{2}} \cos \phi$$

Equating the red pieces:

$$P_{avg} = I_{rms}^2 R$$

Power in AC Circuits

$$P_{\text{avg}} = I_{\text{rms}} \Delta V_{\text{rms}} \cos \phi$$

The average power delivered is largest when $\phi = 0$. Current is in phase with the voltage.

Then:

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Then:

$$P_{\text{avg}} = I_{\text{rms}} \Delta V_{\text{rms}}$$

Using our expression for ϕ:

$$\phi = \tan^{-1}\left(\frac{X}{R}\right) = 0 \Rightarrow X = 0$$

Notice that since, $X = 0$:

$$Z = \sqrt{R^2 + 0} = R$$

This is the minimum possible value for Z.

Resonance and Power in AC Circuits

Current is in phase with the voltage when $X = 0$.

$$\begin{aligned} X_L &= X_C \\ \omega L &= \frac{1}{\omega C} \\ \omega &= \frac{1}{\sqrt{LC}} \end{aligned}$$

Resonance and Power in AC Circuits

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If the circuit is driven by an oscillating voltage with frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

the power delivered will be high. This is the **resonance frequency** of the circuit.

Resonance and Power in AC Circuits

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$$\omega_0 = \frac{1}{\sqrt{LC}}$$

the power delivered will be high. This is the **resonance frequency** of the circuit. (Where did we see this before?)

Resonance and Power in AC Circuits

We can express the average power in terms of ΔV_{rms} and Z instead of I_{rms}, which varies with frequency, ω.

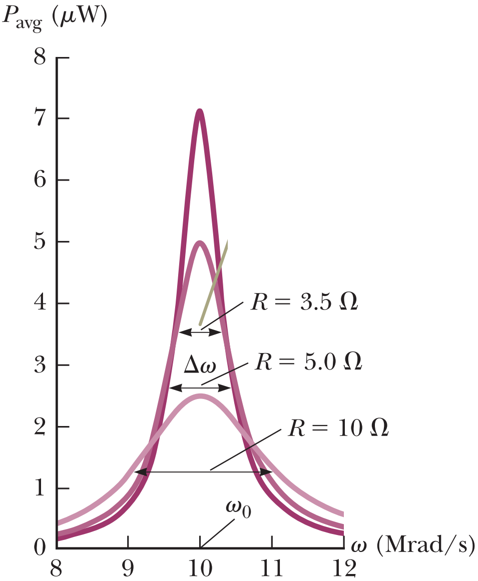
$$P_{\text{avg}} = I_{\text{rms}}^2 R = \left(\frac{\Delta V_{\text{rms}}}{Z}\right)^2 R$$

Using $Z = \sqrt{R^2 + (X_L - X_C)^2}$ and our definitions of X_L, X_C, and ω_0:

$$P_{\text{avg}} = \frac{(\Delta V_{\text{rms}})^2 R \omega^2}{R^2\omega^2 + L^2(\omega^2 - \omega_0^2)^2}$$

This is a Lorentzian function.

Resonance and Power in AC Circuits



Maxwell's Laws

Amazingly, we can summarize the majority of the relations that we have talked about in this course in a set of just 4 equations.

These are together called Maxwell's equations.

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon}$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} + \mu_0 I_{\text{enc}}$$

Gauss's Law for Magnetic Fields

The first of Maxwell's equations is Gauss's Law for E-fields:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon}$$

The second is for Gauss's Law for B-fields:

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

Faraday's Law of Induction

Faraday's Law of Induction is the third of Maxwell's laws.

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

This tells us that a changing magnetic field will induce an electric field.

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But what about the reverse? A changing electric field inducing a magnetic field?

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But what about the reverse? A changing electric field inducing a magnetic field?

It does happen!

Maxwell's Law of Induction

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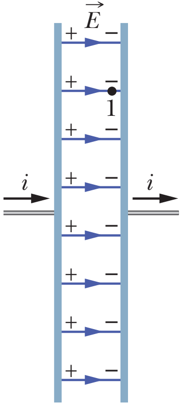
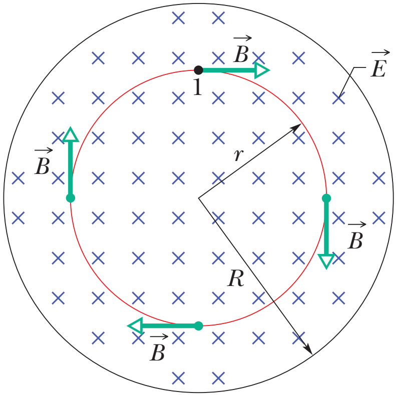




Ampere-Maxwell Law

However, a changing electric field is not the only cause of a magnetic field.

We know from Ampere's Law:

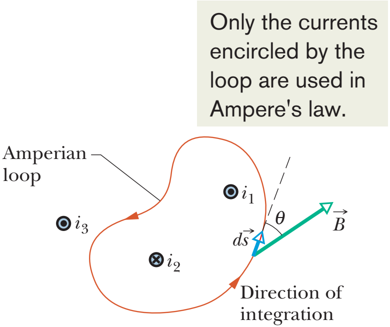
$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 i_{\text{enc}}$$

that a moving charge (current) causes a magnetic field also.

Reminder: Ampère's Law

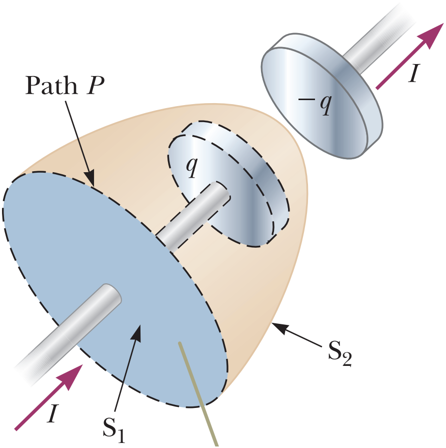
$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{enc}$$

The line integral of the magnetic field around a closed loop is proportional to the current that flows *through* the loop.[1]



[1]That is, the current that flows through **any surface bounded by the loop.**

Maxwell's Law of Induction



Surfaces S_1 and S_2 have different currents flowing through them!

Maxwell's Law of Induction

Maxwell realized that there should be another term in Ampère's law.

He introduced the notion of a **displacement current**:

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

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He introduced the notion of a **displacement current**:

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

Note: The displacement "current" is not a current and has nothing to do with displacement. However, it does have units of Amps.

This completes Ampere's law as:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 (I_{enc} + I_d)$$

The Ampère-Maxwell's Law

The fourth and last of Maxwell's equations:

The Ampère-Maxwell's Law

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} + \mu_0 I_{enc}$$

Differential form:

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

Summary

- RLC series circuit
- impedance
- power
- resonance

Collected Homework 4! due tomorrow.

Final Exam Tuesday, Mar 27, 9:15-11:15am, S35 (here).

Homework Serway & Jewett:

- PREV: Ch 33, onward from page 1021. Problems: 9, 15, 19
- NEW: Ch 33, onward from page 1021. Problems: 25, 27, 33, 43, 45, 65