2. **[15pts]** Initially, a container of volume V contains n moles of helium gas at a pressure P. Suppose the pressure is of the gas is increased to $4P$ while the volume is unchanged.

   (a) By what factor does the kinetic energy of the helium atoms change? [4 pts]

   (b) By what factor does the root-mean-square (rms) speed of the atoms change? [2 pts]

   (c) By what factor does the most probable speed of the atoms change? [2 pts]

   (d) On the same (labeled) axes, sketch the distribution of molecular speeds for each pressure. Label the most probable speed and the rms speed on each distribution. [7 pts]

a) Helium is a monatomic gas : total KE = translational KE

total $K = \frac{3}{2} N k_B T = \frac{3}{2} n RT$ $PV = nRT$ or $PV = N k_B T$

$$K = \frac{3}{2} PV$$

$P \to 4P$

$\Rightarrow K \to K' = \frac{3}{2}(4P)V$

$= \underline{4K}$ Increases by a factor of 4.

b) $v_{rms} = \sqrt{\frac{3 k_B T}{m_0}} = \sqrt{\frac{3RT}{M}}$

$\Rightarrow v_{rms} = \sqrt{\frac{3PV}{N m_0}} = \sqrt{\frac{3PV}{nM}}$

$P \to 4P$

$\Rightarrow v_{rms} \to v_{rms}' = \sqrt{\frac{3(4P)V}{N m_0}} = 2\sqrt{\frac{3PV}{N m_0}}$

$= \underline{2 v_{rms}}$

Increases by a factor of 2.

c) $v_{mp} = \sqrt{\frac{2 k_B T}{m_0}}$

$= \sqrt{\frac{2PV}{N m_0}}$

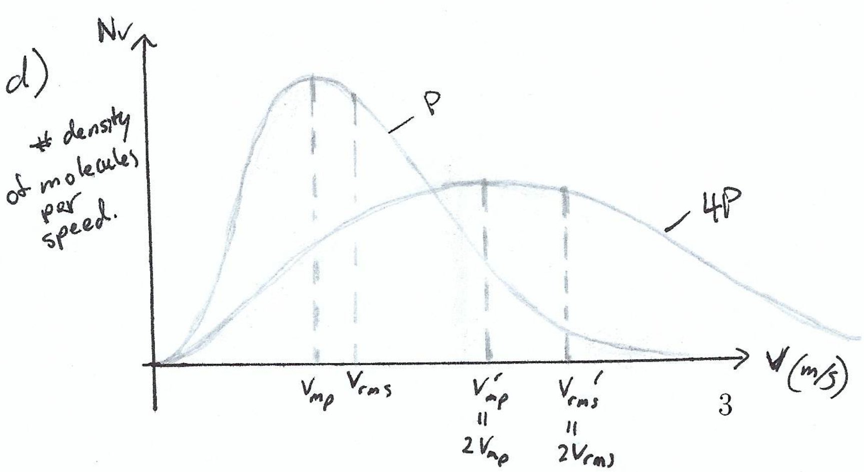
$P \to 4P$

$\Rightarrow v_{mp} \to v_{mp}' = \sqrt{\frac{2(4P)V}{N m_0}}$

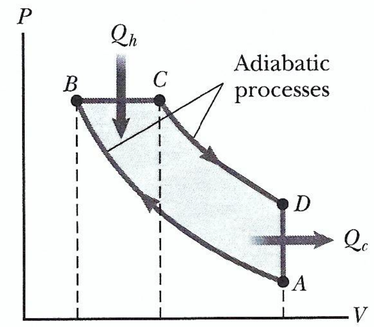
$= \underline{2 v_{mp}}$

Increases by a factor of 2.

d)



3. **[23pts]** An idealized diesel engine operates in the cycle shown. Fuel is sprayed into the cylinder at the point of maximum compression, B. Combustion occurs during the expansion $B \to C$, which is modeled as an isobaric process. The gas in the cycle has n moles, a molar heat capacity at constant pressure of C_p, and a molar heat capacity at constant volume of C_v, and ratio γ. T_A, T_B, T_C, and T_D are the temperatures at points A, B, C, and D, respectively.



(a) What is the heat transferred to the gas in processes C $\to$ D and A $\to$ B? [2 pts]

(b) Show that the efficiency of an engine operating in this cycle is [5 pts]

$$e = 1 - \frac{1}{\gamma}\left(\frac{T_D - T_A}{T_C - T_B}\right)$$

(c) What is the entropy change in the process B $\to$ C? [3 pts]

(d) What is the entropy change in the process D $\to$ A? [3 pts]

(e) Show that [5 pts]

$$\frac{T_D}{T_A} = \left(\frac{T_C}{T_B}\right)^{\gamma}$$

(f) Using your results from (d), (e), and ~~(f)~~ (c), find/confirm the net change in entropy of the working fluid going clockwise around the cycle from A $\to$ A. [5 pts]

a) $Q = 0$ for both, processes are adiabatic

b) $e = 1 - \frac{|Q_c|}{|Q_h|}$

$Q_c = nC_v \Delta T_{DA}$ (is -ve)

$Q_h = nC_p \Delta T_{BC}$ (is +ve)

$e = 1 - \frac{\cancel{n} C_v (T_D - T_A)}{\cancel{n} C_p (T_C - T_B)}$ $\quad \gamma = \frac{C_p}{C_v}$

$\underline{e = 1 - \frac{1}{\gamma}\left(\frac{T_D - T_A}{T_C - T_B}\right)}$

c) $\Delta S = \int_i^f \frac{dQ_r}{T}$

$dQ = nC_p dT$

$\Delta S_{BC} = nC_p \int_{T_B}^{T_C} \frac{dT}{T}$

$\underline{\Delta S_{BC} = nC_p \ln\left(\frac{T_C}{T_B}\right)}$

d) $dQ = nC_v dT$

$\Delta S_{DA} = nC_v \int_{T_D}^{T_A} \frac{dT}{T}$

$\underline{\Delta S_{DA} = nC_v \ln\left(\frac{T_A}{T_D}\right)}$

e) Adiabatic process : $TV^{\gamma-1} = \text{const.}$

$$CD: \quad T_D V_D^{\gamma-1} = T_C V_C^{\gamma-1} \quad \text{— (1)}$$

$$AB: \quad T_A V_A^{\gamma-1} = T_B V_B^{\gamma-1} \quad \text{— (2)}$$

(1) ÷ (2) and noting $V_D = V_A$:

$$\frac{T_D}{T_A} = \frac{T_C}{T_B}\left(\frac{V_C}{V_B}\right)^{\gamma-1}$$

$$BC: \quad P_C V_C = nRT_C \quad \text{— (3)}$$

and $P_B V_B = nRT_B$ — (4)

(3) ÷ (4) and noting $P_B = P_C$:

$$\frac{V_C}{V_B} = \frac{T_C}{T_B}$$

$$\frac{T_D}{T_A} = \frac{T_C}{T_B}\left(\frac{T_C}{T_B}\right)^{\gamma-1}$$

$$\underline{\frac{T_D}{T_A} = \left(\frac{T_C}{T_B}\right)^{\gamma}}$$

f) expect $\Delta S_{cycle} = 0$

$$\Delta S_{cycle} = \cancel{\Delta S_{AB}}^{\,0} + \Delta S_{BC} + \cancel{\Delta S_{CD}}^{\,0 \text{ adiabatic.}} + \Delta S_{DA}$$

$$= nC_p \ln\left(\frac{T_C}{T_B}\right) + nC_v \ln\left(\frac{T_A}{T_D}\right)$$

using (e)

$$= nC_p \ln\left(\frac{T_C}{T_B}\right) + nC_v \ln\left[\left(\frac{T_B}{T_C}\right)^{\gamma}\right]$$

$$= nC_p \ln\left(\frac{T_C}{T_B}\right) + n\underbrace{C_v\gamma}_{C_p} \ln\left(\frac{T_B}{T_C}\right) = nC_p\left(\ln\left(\frac{T_C}{T_B}\right) - \ln\left(\frac{T_C}{T_B}\right)\right)$$

$$\underline{= 0}$$

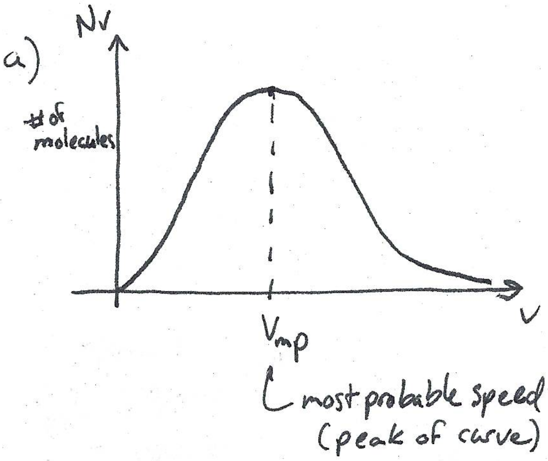
3. Consider a gas of identical molecules, each with mass m_0, in thermal equilibrium at temperature, T.

   (a) Sketch the distribution of molecular speeds and label the most probable speed for a gas molecule on it. [4 pts]

   (b) Starting from the Maxwell-Boltzmann speed distribution, find an expression for the most probable speed of a gas molecule. [5 pts]

   (c) What is the the root-mean-square (rms) speed for the gas and how does the rms speed relate to the most probable speed? [2 pts]

a)



b) $$N_v = \underbrace{4\pi N \left(\frac{m_0}{2\pi k_B T}\right)^{3/2}}_{\text{const.} = c} v^2 e^{-\frac{m_0 v^2}{2k_B T}}$$

most probable speed occurs at peak of distribution. Occurs when:

$$\frac{dN_v}{dv} = 0$$

$$\frac{dN_v}{dv} = c\left(2v\,\cancel{e^{-\frac{m_0 v^2}{2k_BT}}} + v^2\left(-\frac{m_0(2v)}{2k_BT}\,\cancel{e^{-\frac{m_0 v^2}{2k_BT}}}\right)\right) = 0$$

$$\cancel{2}\cancel{v} = \frac{\cancel{2}m_0 v^{\cancel{3}\,2}}{2k_BT}$$

$$v^2 = \frac{2k_BT}{m_0}$$

$$\underline{V_{mp} = \sqrt{\frac{2k_BT}{m_0}}}$$

c) $$\underline{V_{rms} = \sqrt{\overline{V^2}} = \sqrt{\frac{3k_BT}{m_0}}}$$

$$\left(\begin{aligned} K = \tfrac{1}{2} m_0 \overline{V^2} &= \tfrac{3}{2}k_BT \\ \overline{V^2} &= \frac{3k_BT}{m_0} \\ \sqrt{\overline{V^2}} &= \sqrt{\frac{3k_BT}{m_0}} \end{aligned} \right)$$

The RMS speed is greater than the most probable speed.

4. A sample of a diatomic ideal gas with specific heat ratio 1.40, confined to a cylinder, is carried through a closed cycle. The gas is initially at pressure P_0 and volume V_0. First, in step 1, its pressure is tripled under constant volume. Then, in step 2, it expands adiabatically to its original pressure. Finally, in step 3, the gas is compressed isobarically to its original volume.

   (a) Draw a PV diagram of this cycle, labeling the steps. [4 pts]

   (b) Find an expression for the volume of the gas at the end of the adiabatic expansion in terms of V_0. [3 pts]

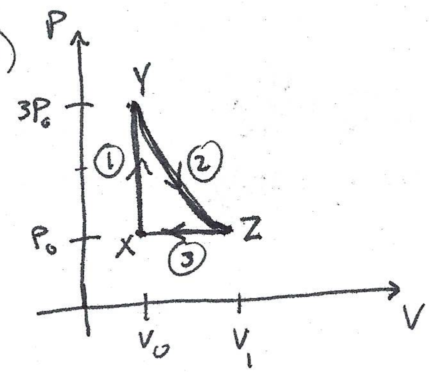
   (c) Assuming that there are n moles of gas in the sample, find expressions for the temperatures of the gas at the start of the adiabatic expansion, at the end of the expansion, and at the end of the cycle. [4 pts]

   (d) During which of the three steps does heat enter the gas? During which of the three steps does heat leave the gas? [2 pts]

   (e) If the cycle is used as a heat engine, what is the engine's efficiency? (Find a numerical value.) [7 pts]

   (f) How does that compare to the efficiency of a Carnot engine operating between the same temperature extremes? (Find the Carnot engine's efficiency.) [3 pts]

a)



$\gamma = 1.40 = \frac{7}{5}$

b) At the start of the adiabatic expansion (point Y): $3P_0, V_0$

adiab: $P_i V_i^\gamma = P_f V_f^\gamma$

$(3\cancel{P_0})(V_0)^\gamma = \cancel{P_0} V_1^\gamma$

$V_1^\gamma = 3V_0^\gamma$

$V_1 = 3^{1/\gamma} V_0$

c) $PV = nRT$

start of expansion (point Y)

$3P_0V_0 = nRT_Y$

$T_Y = \frac{3P_0V_0}{nR}$

end of expansion (point Z)

$P_0(3^{1/\gamma}V_0) = nRT_Z$

$T_Z = \frac{3^{1/\gamma}P_0V_0}{nR}$

end of cycle (point X)

$P_0V_0 = nRT_X$

$T_X = \frac{P_0V_0}{nR}$

d) Heat enters the gas during step 1.
($\Delta E_{int,1}$ is positive, $W_1 = 0$, $\Rightarrow Q_1 > 0$)

Heat leaves the gas in step 3.
($\Delta E_{int,3}$ is negative, $W_3 > 0 \Rightarrow Q_3 < 0$)
($Q_2 = 0$, adiabatic)

e) $e = 1 - \frac{|Q_C|}{|Q_H|}$

const. pressure process

$Q_C = Q_3 = nC_p\Delta T$

$= n(\frac{7}{2}R)(T_X - T_Z)$

$= \frac{7\cancel{n}\cancel{R}}{2}\left(\frac{P_0V_0}{\cancel{nR}} - \frac{3^{1/\gamma}P_0V_0}{\cancel{nR}}\right)$

$= -\frac{7}{2}(3^{1/\gamma} - 1)P_0V_0$ 5

const. vol. process

$Q_H = Q_1 = nC_v\Delta T$

$= n(\frac{5}{2}R)(T_Y - T_X)$

$= \frac{5\cancel{n}\cancel{R}}{2}\left(\frac{3P_0V_0}{\cancel{nR}} - \frac{P_0V_0}{\cancel{nR}}\right)$

$= 5\,P_0V_0$

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e) cont'd.

$$e = 1 - \frac{\left| -\frac{7}{2}\left(3^{1/\gamma} - 1\right) P_0 V_0 \right|}{\left| \frac{5}{\cancel{2}} P_0 V_0 \right|}$$

$$= 1 - \frac{\frac{7}{2}\left(\left(3^{1/\gamma}\right) - 1\right) \cancel{P_0 V_0}}{5 \cancel{P_0 V_0}}$$

$$= 1 - 0.83426$$

$$= 0.166 \qquad (3\ \text{sf.})$$

$$\underline{e = 16.6\%}$$

f) Carnot engine

$$e_c = 1 - \frac{T_c}{T_H}$$

Coldest temperature in the cycle is $T_x = \frac{P_0 V_0}{nR}$

hottest " " " " is $T_y = \frac{3 P_0 V_0}{nR}$

$$= 1 - \frac{\frac{\cancel{P_0 V_0}}{\cancel{nR}}}{\frac{3 \cancel{P_0 V_0}}{\cancel{nR}}}$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

$$\underline{e_c = 66.7\%}$$

The Carnot engine is much more efficient. (More than 4× more efficient.)