Waves
Pulse Propagation
The Wave Equation
Sine Waves

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May 17, 2017
Last time

- oscillations
- simple harmonic motion (SHM)
- spring systems
- energy in SHM
- pendula
- introducing waves
- kinds of waves
Overview

- wave speed on a string
- pulse propagation
- the wave equation
- solutions to the wave equation
- sine waves
- transverse speed and acceleration
Wave Motion

Wave

A disturbance or oscillation that transfers energy through matter or space.

The waveform moves along the medium and energy is carried with it.

The particles in the medium *do not* move along with the wave.

The particles in the medium are briefly shifted from their equilibrium positions, and then return to them.
Wave Speed on a String

How fast does a disturbance propagate on a string under tension?

As the pulse moves along the string, new elements of the string are displaced from their equilibrium positions.
Wave Speed on a String

Imagine traveling with the pulse at speed \( v \) to the right. Each small section of the rope travels to the left along a circular arc from your point of view.

We will find how fast a point on the string moves backwards relative to the wave pulse.
Wave Speed on a String

We can use the force diagram to find the force on a small length of string $\Delta s$:

$$F_{\text{net}} = 2T \sin \theta \approx 2T \theta$$
Wave Speed on a String

Consider the centripetal force on the piece of string.

If $R$ is the radius of curvature and $m$ is the mass of the small piece of string:

$$F_{\text{net}} = \frac{mv^2}{R}$$
Wave Speed on a String

Consider the centripetal force on the piece of string.

If \( R \) is the radius of curvature and \( m \) is the mass of the small piece of string:

\[
F_{\text{net}} = \frac{mv^2}{R}
\]

Suppose the string has mass density \( \mu \) (units: kg m\(^{-1}\))

\[
m = \mu \Delta s = \mu R(2\theta)
\]

Put this into our expression for centripetal force:

\[
F_{\text{net}} = \frac{2\mu R\theta v^2}{R}
\]
Wave Speed on a String

Put this into our expression for centripetal force:

$$F_{net} = 2\mu \theta v^2$$

And using eq. (1), $F_{net} = 2T\theta$:

$$2T\theta = 2\mu \theta v^2$$

The wave speed is:

$$v = \sqrt{\frac{T}{\mu}}$$

For a given string under a given tension, all waves travel with the same speed!
If you stretch a rubber hose and pluck it, you can observe a pulse traveling up and down the hose.

What happens to the speed of the pulse if you stretch the hose more tightly?

(A) It increases.
(B) It decreases.
(C) It is constant.
(D) It changes unpredictably.

---

1Serway & Jewett, page 499, objective question 2.
If you stretch a rubber hose and pluck it, you can observe a pulse traveling up and down the hose.

What happens to the speed if you fill the hose with water?

(A) It increases.
(B) It decreases.
(C) It is constant.
(D) It changes unpredictably.

---

1Serway & Jewett, page 499, objective question 2.
Example

A uniform string has a mass of $m_s$ and a length of $\ell$. The string passes over a pulley and supports an block of mass $m_b$. Find the speed of a pulse traveling along this string. (Assume the vertical piece of the rope is very short.)
Example

A uniform string has a mass of $m_s$ and a length of $\ell$. The string passes over a pulley and supports an block of mass $m_b$. Find the speed of a pulse traveling along this string. (Assume the vertical piece of the rope is very short.)

$$v = \sqrt{\frac{m_b g \ell}{m_s}}$$
A wave pulse (in a plane) at a moment in time can be described in terms of \( x \) and \( y \) coordinates, giving \( y(x) \).

We know that the pulse will move with speed \( v \) and be displaced, say in the positive \( x \) direction, while maintaining its shape.

That means we can also give \( y \) as a function of time, \( y(x, t) \).

Consider a moving reference frame, \( S' \), with the pulse at rest, \( y'(x') = f(x') \), no time dependence. Galilean transformation:

\[
x' = x - vt
\]
Pulse Propagation

\[ x' = x - vt \]

Then in the rest-frame of the string

\[ y(x, t) = f(x') = f(x - vt) \]
The shape of the pulse is given by $f(x)$ and can be arbitrary.

Whatever the form of $f$, if the pulse moves in the $+x$ direction:

$$y(x, t) = f(x - vt)$$

If the pulse moves in the $-x$ direction:

$$y(x, t) = f(x + vt)$$
Wave Pulse Example 16.1

A pulse moving to the right along the x axis is represented by the wave function

\[ y(x, t) = \frac{2}{(x - 3.0t)^2 + 1} \]

where \( x \) and \( y \) are measured in centimeters and \( t \) is measured in seconds.

What is the wave speed?

Find expressions for the wave function at \( t = 0 \), \( t = 1.0 \) s, and \( t = 2.0 \) s.

\footnote{Serway & Jewett, page 486.}
\footnote{This function is an unnormalized Cauchy distribution, or as physicists say “it has a Lorentzian profile”.
}
Wave Pulse Example 16.1

A pulse moving to the right along the x axis is shown in Figure 16.6a, where $y(x,0)$ represents a pulse with a similar shape as that in Figure 16.6, but moving to the left as time progresses.

**What if?**

Write the wave function expression at $t = 5$ s.

A more general wave function is given by:

$$y(x,t) = \frac{1}{2} \left[ \text{sine function} \right]$$

For each of these expressions, we can substitute various values of $x$ and $t$ to see how the wave function changes.

**Quick Quiz**

Consider “the wave” at a baseball game: people stand up and raise their arms as the wave arrives at each location. As each person steps forward, the gap moves through the line. Is the propagation of this gap (a) transverse or (b) longitudinal?

**Solution**

We categorize this example as a relatively simple analysis of wave propagation.

**Conceptualize**

For each of the following, determine whether the wave is transverse or longitudinal.

(a) A pulse moving to the right along the x axis is given by

$$y(x,t) = \frac{1}{2} \left[ \text{sine function} \right]$$

(b) A pulse moving to the right along the x axis is given by

$$y(x,t) = \frac{1}{2} \left[ \text{cosine function} \right]$$

For each of these expressions, we can substitute various values of $x$ and $t$ to see how the wave function changes.

**Finalize**

We can analyze the wave function at different times to see how the pulse moves. For example,

$$y(x,t) = \frac{1}{2} \left[ \text{sine function} \right]$$

At $t = 5$ s, we find that the pulse has moved to the left. How would that change the situation?

**Figure 16.6**

- **a**: $y(x,0)$
- **b**: $y(x,1.0)$
- **c**: $y(x,2.0)$
The Wave Equation

Can we find a general equation describing the displacement \((y)\) of our medium as a function of position \((x)\) and time?

Start by considering a string carrying a disturbance.
The Wave Equation

Consider a small length of string $\Delta x$.

As we did for oscillations, start from Newton’s 2nd law.

\[ F_y = ma_y \]

\[ T \sin \theta_B - T \sin \theta_A = (\mu \Delta x) \frac{\partial^2 y}{\partial t^2} \]

For small angles

\[ \sin \theta \approx \tan \theta \]
The Wave Equation

We can write $\tan \theta$ as the slope of $y(x)$:

$$\tan \theta = \frac{\partial y}{\partial x}$$

Now Newton’s second law becomes:

$$T \left( \left. \frac{\partial y}{\partial x} \right|_{x=B} - \left. \frac{\partial y}{\partial x} \right|_{x=A} \right) = (\mu \Delta x) \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} = \frac{\left. \frac{\partial y}{\partial x} \right|_{x=B} - \left. \frac{\partial y}{\partial x} \right|_{x=A}}{\Delta x}$$
The Wave Equation

We can write \( \tan \theta \) as the slope of \( y(x) \):

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\tan \theta = \frac{\partial y}{\partial x}
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\[
\frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} = \frac{\frac{\partial y}{\partial x} \bigg|_{x=B} - \frac{\partial y}{\partial x} \bigg|_{x=A}}{\Delta x}
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We need to take the limit of this expression as we consider an infinitesimal piece of string: \( \Delta x \to 0 \).
The Wave Equation

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Now Newton’s second law becomes:

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T \left( \frac{\partial y}{\partial x} \bigg|_{x=B} - \frac{\partial y}{\partial x} \bigg|_{x=A} \right) = (\mu \Delta x) \frac{\partial^2 y}{\partial t^2}
\]

\[
\mu \frac{\partial^2 y}{T \partial t^2} = \frac{\frac{\partial y}{\partial x} \bigg|_{x=B} - \frac{\partial y}{\partial x} \bigg|_{x=A}}{\Delta x}
\]

We need to take the limit of this expression as we consider an infinitesimal piece of string: \( \Delta x \to 0 \).

\[
\mu \frac{\partial^2 y}{T \partial t^2} = \frac{\partial^2 y}{\partial x^2}
\]

where we use the definition of the partial derivative.
The Wave Equation

\[
\frac{\mu \frac{\partial^2 y}{\partial t^2}}{T \frac{\partial^2 y}{\partial x^2}} = \frac{\partial^2 y}{\partial x^2}
\]

Remember that the speed of a wave on a string is

\[
v = \sqrt{\frac{T}{\mu}}
\]

The wave equation:

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

Even though we derived this for a string, it applies much more generally!
The Wave Equation

We can model longitudinal waves like sound waves by a series of masses connected by springs, length $h$.

\[ u(x), u(x+h), u(x+2h) \]

\[ u_1, u_2, u_3 \]

$u$ is a function that gives the displacement of the mass at each equilibrium position $x$, $x + h$, etc.

For such a case, the propagation speed is

\[ v = \sqrt{\frac{KL}{\mu}} \]

where $K$ is the spring constant of the entire spring chain, $L$ is the length, and $\mu$ is the mass density.

\[ ^1 \text{Figure from Wikipedia, by Sebastian Henckel.} \]
The Wave Equation

$u$ is a function that gives the displacement of the mass at each equilibrium position $x$, $x + h$, etc.

Consider the mass, $m$, at equilibrium position $x + h$

$$F = ma$$

$$k(u_3 - u_2) - k(u_2 - u_1) = m \frac{\partial^2 u}{\partial t^2}$$

$$\frac{m}{k} \frac{\partial^2 u}{\partial t^2} = u_3 - 2u_2 + u_1$$

---

\(^1\text{Figure from Wikipedia, by Sebastian Henckel.}\)
The Wave Equation

\[ m \frac{\partial^2 u}{k \partial t^2} = u_3 - 2u_2 + u_1 \]

We can re-write \( \frac{m}{k} \) in terms of quantities for the entire spring chain. Suppose there are \( N \) masses.

\( m = \frac{\mu L}{N} \) and \( k = NK \) and \( N = \frac{L}{h} \)

\[ \frac{\mu}{KL} \frac{\partial^2 u}{\partial t^2} = \frac{u(x + 2h) - 2u(x + h) + u(x)}{h^2} \]

Letting \( N \to \infty \) and \( h \to 0 \), the RHS is the definition of the 2nd derivative. Same equation!

\[ \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \]
Solutions to the Wave Equation

Earlier we reasoned that a function of the form:

\[ y(x, t) = f(x - vt) \]

should describe a propagating wave pulse.

Does it satisfy the wave equation?

\[ \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \]
Solutions to the Wave Equation

Earlier we reasoned that a function of the form:

$$y(x, t) = f(x - vt)$$

should describe a propagating wave pulse.

Does it satisfy the wave equation?

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Let $u = x - vt$, so we can use the chain rule:

$$\frac{\partial y}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial y}{\partial u} = (1) \frac{\partial y}{\partial u}$$

and

$$\frac{\partial y}{\partial t} = \frac{\partial u}{\partial t} \frac{\partial y}{\partial u} = -v \frac{\partial y}{\partial u}$$
Solutions to the Wave Equation

Replacing $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ in the wave equation:

$$\frac{\partial^2 y}{\partial u^2} = \frac{1}{v^2(v^2)} \frac{\partial^2 y}{\partial u^2}$$

$$1 = 1$$

The LHS does equal the RHS!

$y(x, t) = f(x - vt)$ is a solution to the wave equation for any (well-behaved) function $f$. 
Sine Waves

An important form of the function $f$ is a sine or cosine wave. (All called “sine waves”).

This is the simplest periodic, continuous wave.

It is also the wave that is formed by a (driven) simple harmonic oscillator connected to the medium.
Wave Quantities

- Wavelength ($\lambda$)
- Amplitude
- Period ($T$)
- Distance
- Time
Recall, the definition of frequency, from period $T$:

$$f = \frac{1}{T}$$

and

$$\omega = \frac{2\pi}{T} = 2\pi f$$

We also define a new quantity.

**Wave number, $k$**

$$k = \frac{2\pi}{\lambda}$$

units: $m^{-1}$
Wave speed

How fast does a wave travel?

\[ \text{speed} = \frac{\text{distance}}{\text{time}} \]

It travels the distance of one complete cycle in the time for one complete cycle.

\[ v = \frac{\lambda}{T} \]

But since frequency is the inverse of the time period, we can relate speed to frequency and wavelength:

\[ v = f\lambda \]
Wave speed

\[ v = f \lambda \]

Since \( \omega = 2\pi f \) and \( k = \frac{2\pi}{\lambda} \):

\[ v = \frac{\omega}{k} \]
In this section, we introduce an important analysis model of a traveling sinusoidal wave. A snapshot of a sinusoidal wave is shown in Figure 16.8a. The position of one element of the medium as a function of time is shown in Figure 16.8b.

The distance from one crest to the next is called the wavelength, \( \lambda \). In general, the frequency of a periodic wave is the number of cycles per unit time interval. The frequency, \( f \), is related to the period, \( T \), by

\[
f = \frac{1}{T}
\]

The distance between adjacent crests or adjacent troughs is called the wavelength, \( \lambda \). The period of the wave is the same as the time required for the wave to travel one wavelength. A sinusoidal wave can be combined to build complex waves, just as we combined particles. Therefore, the particle can be considered a basic building block. An ideal particle has zero size. We can build physical objects with nonzero size as combinations of particles.

In what follows, we will develop the principal features and mathematical representations of the analysis model of a traveling sinusoidal wave. This model is used in situations in which a wave moves through space without interacting with other waves or particles.

The sinusoidal wave is the simplest example of a periodic continuous wave and is formulated as a function of motion that can occur. First, the entire waveform in Figure 16.7 moves to the right with a speed \( v \) through space. Figure 16.7 represents a snapshot of a traveling sinusoidal wave at time \( t = 0 \). The wave represented by this curve is called a sinusoidal wave, because

\[
y(x, t) = A \sin \left( \frac{2\pi}{\lambda} (x - vt) \right)
\]

This is usually written in a slightly different form...
Sine Waves

\[ y(x, t) = A \sin (kx - \omega t + \phi) \]

where \( \phi \) is a phase constant.
Quick Quiz 16.2\textsuperscript{1} A sinusoidal wave of frequency $f$ is traveling along a stretched string. The string is brought to rest, and a second traveling wave of frequency $2f$ is established on the string.

What is the wave speed of the second wave?

(A) twice that of the first wave  
(B) half that of the first wave  
(C) the same as that of the first wave  
(D) impossible to determine

\textsuperscript{1}Serway & Jewett, page 489.
Quick Quiz 16.2\(^1\) A sinusoidal wave of frequency \(f\) is traveling along a stretched string. The string is brought to rest, and a second traveling wave of frequency \(2f\) is established on the string.

What is the wavelength of the second wave?

(A) twice that of the first wave  
(B) half that of the first wave  
(C) the same as that of the first wave  
(D) impossible to determine

\(^1\)Serway & Jewett, page 489.
Question

Quick Quiz 16.2 A sinusoidal wave of frequency $f$ is traveling along a stretched string. The string is brought to rest, and a second traveling wave of frequency $2f$ is established on the string. What is the amplitude of the second wave?

(A) twice that of the first wave
(B) half that of the first wave
(C) the same as that of the first wave
(D) impossible to determine

---

Serway & Jewett, page 489.
Consider a point, \( P \), on a string carrying a sine wave.

Suppose that point is at a fixed horizontal position \( x = 5\lambda/4 \), a constant.

The \( y \) coordinate of \( P \) varies as:

\[
y \left( \frac{5\lambda}{4}, t \right) = A \sin(-\omega t + 5\pi/2)
\]

\[
= A \cos(\omega t)
\]

The point is in simple harmonic motion!
Sine waves: Transverse Speed and Transverse Acceleration

The transverse speed $v_y$ is the speed at which a single point on the medium (string) travels perpendicular to the propagation direction of the wave.

We can find this from the wave function

$$y(x, t) = A \sin(kx - \omega t)$$
Sine waves: Transverse Speed and Transverse Acceleration

The transverse speed $v_y$ is the speed at which a single point on the medium (string) travels perpendicular to the propagation direction of the wave.

We can find this from the wave function

$$y(x, t) = A \sin(kx - \omega t)$$

$$v_y = \frac{\partial y}{\partial t} = -\omega A \cos(kx - \omega t)$$

For the transverse acceleration, we just take the derivative again:

$$a_y = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(kx - \omega t)$$
Sine waves: Transverse Speed and Transverse Acceleration

\[ v_y = -\omega A \cos(kx - \omega t) \]
\[ a_y = -\omega^2 A \sin(kx - \omega t) = -\omega^2 y \]

If we fix \( x = \text{const.} \) these are exactly the equations we had for SHM!

The maximum transverse speed of a point \( P \) on the string is when it passes through its equilibrium position.

\[ v_{y,\text{max}} = \omega A \]

The maximum acceleration occurs when \( y = A \).

\[ a_y = \omega^2 A \]
Can a wave on a string move with a wave speed that is greater than the maximum transverse speed $v_{y,\text{max}}$ of an element of the string?

(A) yes
(B) no
Can the wave speed be much greater than the maximum element speed?

(A) yes
(B) no
Questions

Can the wave speed be equal to the maximum element speed?

(A) yes
(B) no
Can the wave speed be less than $v_{y,\text{max}}$?

(A) yes
(B) no
Summary

- wave speed on a string
- pulse propagation
- the wave equation
- solutions to the wave equation
- sine waves

Homework Serway & Jewett:

- Ch 16, onward from page 499. OQs: 3, 5, 9; CQs: 1, 5, 9; Probs: 1, 3, 5, 9, 11, 19, 23, 29, 41, 43, 53, 59, 60