Waves
Solutions to the Wave Equation
Sine Waves
Transverse Speed and Acceleration

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Last time

- pulse propagation
- the wave equation

Overview

- solutions to the wave equation
- sine waves
- transverse speed and acceleration

Solutions to the Wave Equation

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should describe a propagating wave pulse.

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Does it satisfy the wave equation?

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Solutions to the Wave Equation

Does $y(x, t) = f(x - vt)$ satisfy the wave equation?

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Solutions to the Wave Equation

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Let $u = x - vt$, so we can use the chain rule:

$$\frac{\partial y}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial y}{\partial u} = (1) f_u' \qquad ; \qquad \frac{\partial^2 y}{\partial x^2} = (1^2) f_u''$$

and

$$\frac{\partial y}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial y}{\partial u} = -v\, f_u' \qquad ; \qquad \frac{\partial^2 y}{\partial t^2} = v^2 f_u''$$

where f_u' is the partial derivative of f wrt u.

Solutions to the Wave Equation

Replacing $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ in the wave equation:

$$\begin{aligned} f_u'' &= \frac{1}{v^2}(v^2) f_u'' \\ 1 &= 1 \end{aligned}$$

The LHS does equal the RHS!

$y(x, t) = f(x \pm vt)$ is a solution to the wave equation for any (well-behaved) function f.

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$y(x, t) = f(x \pm vt)$ is a solution to the wave equation for any (well-behaved) function f.

In fact, any solution to the wave equation can be written:

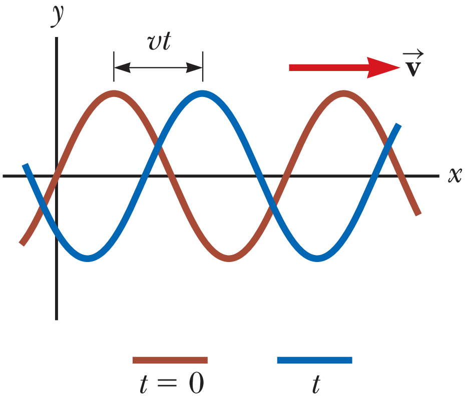
$$y(x, t) = f(x - vt) + g(x + vt)$$

Sine Waves

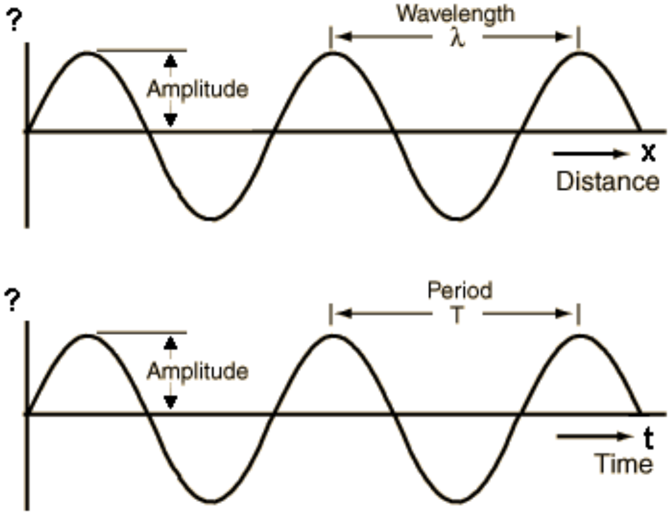
An important form of the function f is a sine or cosine wave. (All called "sine waves"). $y(x, t) = A \sin\left(B(x - vt) + C\right)$

This is the simplest periodic, continuous wave.

It is the wave that is formed by a (driven) simple harmonic oscillator connected to the medium.



Wave Quantities



Wave Quantities

wavelength, λ

the distance from one crest of the wave to the next, or the distance covered by one cycle.
units: length (m)

time period, T

the time for one complete oscillation.
units: time (s)

Sine Waves

Recall, the definition of frequency, from period T:

$$f = \frac{1}{T}$$

and

$$\omega = \frac{2\pi}{T} = 2\pi f$$

We also define a new quantity.

Wave number, k

$$k = \frac{2\pi}{\lambda}$$

units: m^{-1}

Wave speed

How fast does a wave travel?

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

It travels the distance of one complete cycle in the time for one complete cycle.

$$v = \frac{\lambda}{T}$$

But since frequency is the inverse of the time period, we can relate speed to frequency and wavelength:

$$v = f\lambda$$

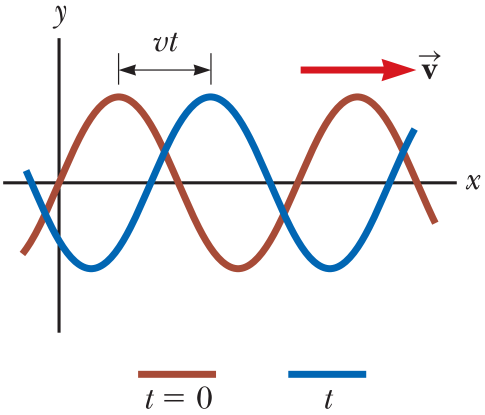
Wave speed

$$v = f\lambda$$

Since $\omega = 2\pi f$ and $k = \frac{2\pi}{\lambda}$:

$$v = \frac{\omega}{k}$$

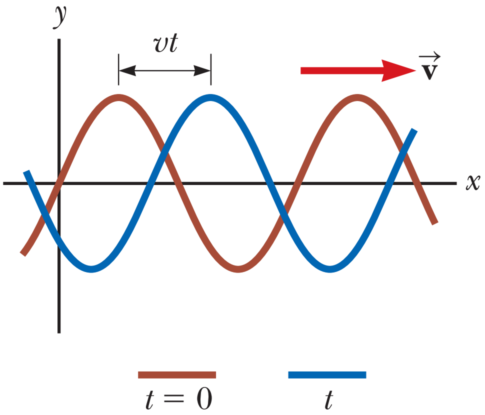
Sine Waves



$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt) + \phi\right)$$

This is usually written in a slightly different form...

Sine Waves



$$y(x,t) = A\sin(kx - \omega t + \phi)$$

where ϕ is a phase constant.

Question

Quick Quiz 16.2[1] A sinusoidal wave of frequency f is traveling along a stretched string. The string is brought to rest, and a second traveling wave of frequency $2f$ is established on the string.

What is the wave speed of the second wave?

(A) twice that of the first wave
(B) half that of the first wave
(C) the same as that of the first wave
(D) impossible to determine

[1] Serway & Jewett, page 489.

Question

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Question

Quick Quiz 16.2[1] A sinusoidal wave of frequency f is traveling along a stretched string. The string is brought to rest, and a second traveling wave of frequency $2f$ is established on the string.

What is the amplitude of the second wave?

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(B) half that of the first wave

(C) the same as that of the first wave

(D) impossible to determine

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Sine waves

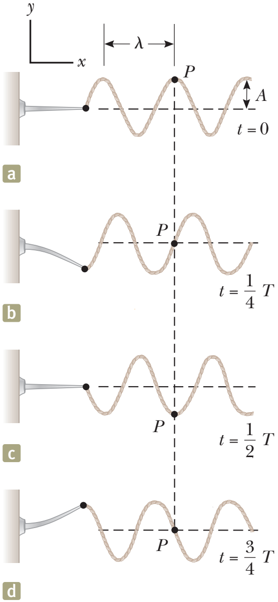
Consider a point, P, on a string carrying a sine wave.

Suppose that point is at a fixed horizontal position $x = 5\lambda/4$, a constant.

The y coordinate of P varies as:

$$
\begin{aligned}
y\left(\frac{5\lambda}{4}, t\right) &= A\sin(-\omega t + 5\pi/2) \\
&= A\cos(\omega t)
\end{aligned}
$$

The point is in simple harmonic motion!



Sine waves: Transverse Speed and Transverse Acceleration

The transverse speed v_y is the speed at which a single point on the medium (string) travels perpendicular to the propagation direction of the wave.

We can find this from the wave function

$$y(x,t) = A\sin(kx - \omega t)$$

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$$y(x,t) = A\sin(kx - \omega t)$$

$$v_y = \frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$$

For the transverse acceleration, we just take the derivative again:

$$a_y = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t)$$

Sine waves: Transverse Speed and Transverse Acceleration

$$v_y = -\omega A \cos(kx - \omega t)$$
$$a_y = -\omega^2 A \sin(kx - \omega t) = -\omega^2 y$$

If we fix $x =$const. these are exactly the equations we had for SHM!

The maximum transverse speed of a point P on the string is when it passes through its equilibrium position.

$$v_{y,\text{max}} = \omega A$$

The maximum acceleration occurs when $y = A$.

$$a_y = \omega^2 A$$

Questions

Can a wave on a string move with a wave speed that is greater than the maximum transverse speed $v_{y,\mathrm{max}}$ of an element of the string?

(A) yes

(B) no

Questions

Can the wave speed be much greater than the maximum element speed?

(A) yes

(B) no

Questions

Can the wave speed be equal to the maximum element speed?

(A) yes

(B) no

Questions

Can the wave speed be less than $v_{y,\mathrm{max}}$?

(A) yes

(B) no

Sine waves: Transverse Speed and Transverse Acceleration

$$v_y = -\omega A \cos(kx - \omega t)$$

$$a_y = -\omega^2 A \sin(kx - \omega t) = -\omega^2 y$$

Summary

- solutions to the wave equation
- sine waves
- transverse speed and acceleration

Homework Serway & Jewett:

- Ch 16, onward from page 499. OQs: 3, 9; CQs: 5; Probs: 5, 9, 11, 19, 41, 43