Waves
Wave Speed on a String
The Wave Equation

Lana Sheridan

De Anza College

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Last time

- oscillations
- simple harmonic motion (SHM)
- spring systems
- introducing waves
- kinds of waves
- pulse propagation

Overview

- wave speed on a string
- the wave equation

Wave Pulse Example 16.1

A pulse moving to the right along the x axis is represented by the wave function

$$y(x,t) = \frac{2}{(x - 3.0t)^2 + 1}$$

where x and y are measured in centimeters and t is measured in seconds.

What is the wave speed? **3.0 cm/s**

Find expressions for the wave function at $t = 0$, $t = 1.0$ s, and $t = 2.0$ s.

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[1]Serway & Jewett, page 486.

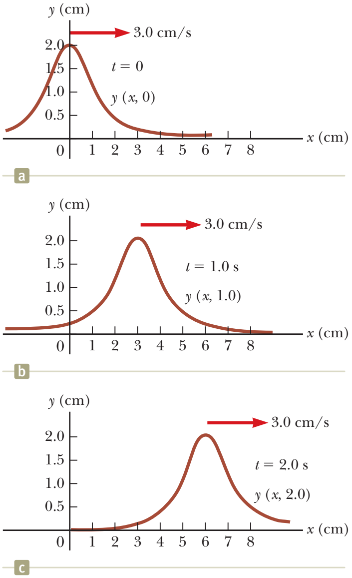
[2]This function is an unnormalized Cauchy distribution, or as physicists say "it has a Lorentzian profile".

Wave Pulse Example 16.1

$$t = 0, \quad y(x,0) = \frac{2}{x^2+1}$$

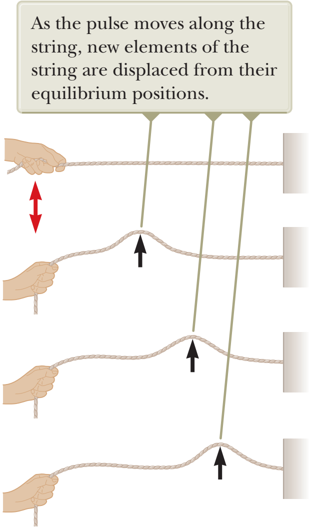
$$t = 1, \quad y(x,1) = \frac{2}{(x-3.0)^2+1}$$

$$t = 2, \quad y(x,2) = \frac{2}{(x-6.0)^2+1}$$



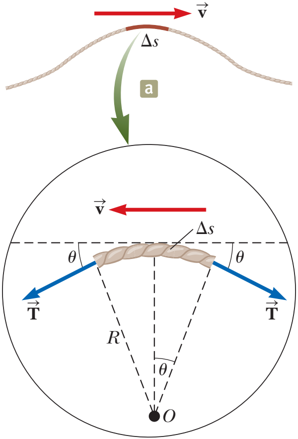
Wave Speed on a String

How fast does a disturbance propagate on a string under tension?



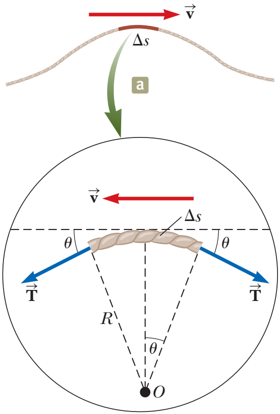
Wave Speed on a String

Imagine traveling with the pulse at speed v to the right. Each small section of the rope travels to the left along a circular arc from your point of view.



We will find find how fast a point on the string *moves backwards relative to the wave pulse*.

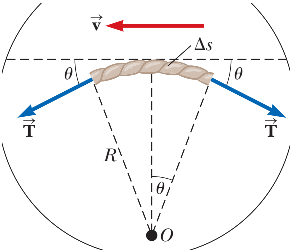
Wave Speed on a String



We can use the force diagram to find the force on a *small* length of string Δs:

$$F_{\text{net}} = 2T \sin \theta \approx 2T\theta \tag{1}$$

Wave Speed on a String



Wave Speed on a String

Consider the centripetal force on the piece of string.

If R is the radius of curvature and m is the mass of the small piece of string:

$$F_{\text{net}} = \frac{mv^2}{R}$$

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Suppose the string has mass density μ (units: kg m^{-1})

$$m = \mu\,\Delta s = \mu R(2\theta)$$

Put this into our expression for centripetal force:

$$F_{\text{net}} = \frac{2\mu R\theta v^2}{R}$$

Wave Speed on a String

Put this into our expression for centripetal force:

$$F_{\text{net}} = 2\mu\theta v^2$$

And using eq. (1), $F_{\text{net}} = 2T\theta$:

$$2T\theta = 2\mu\theta v^2$$

The wave speed is:

$$v = \sqrt{\frac{T}{\mu}}$$

For a given string under a given tension, all waves travel with the same speed!

Wave Speed Question

If you stretch a rubber hose and pluck it, you can observe a pulse traveling up and down the hose.

What happens to the speed of the pulse if you stretch the hose more tightly?

(A) It increases.
(B) It decreases.
(C) It is constant.
(D) It changes unpredictably.

[1] Serway & Jewett, page 499, objective question 2.

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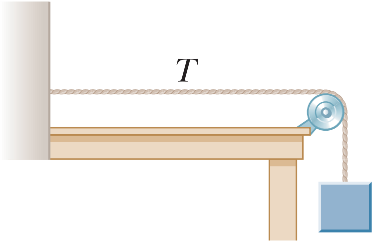
Example

A uniform string has a mass of m_s and a length of ℓ. The string passes over a pulley and supports an block of mass m_b. Find the speed of a pulse traveling along this string. (Assume the vertical piece of the rope is very short.)



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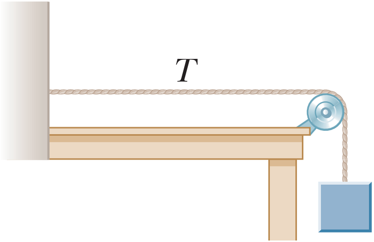
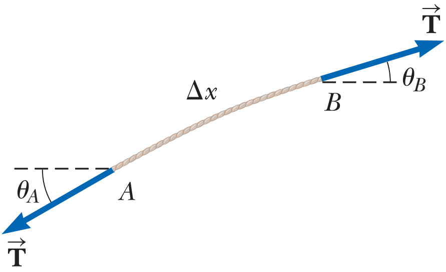


$$v = \sqrt{\frac{m_b g \ell}{m_s}}$$

The Wave Equation

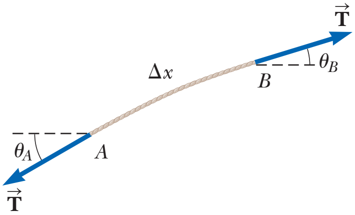
Can we find a general equation describing the displacement (y) of our medium as a function of position (x) and time (t)?

Start by considering a string carrying a disturbance.



The Wave Equation

Consider a small length of string Δx.



As we did for oscillations, start from Newton's 2nd law.

$$\begin{aligned} F_y &= ma_y \\ T\sin\theta_B - T\sin\theta_A &= (\mu\,\Delta x)\frac{\partial^2 y}{\partial t^2} \end{aligned}$$

For small angles

$$\sin\theta \approx \tan\theta$$

The Wave Equation

We can write $\tan\theta$ as the slope of $y(x)$:

$$\tan\theta = \frac{\partial y}{\partial x}$$

Now Newton's second law becomes:

$$\begin{aligned} T\left(\left.\frac{\partial y}{\partial x}\right|_{x=B} - \left.\frac{\partial y}{\partial x}\right|_{x=A} \right) &= (\mu\,\Delta x)\frac{\partial^2 y}{\partial t^2} \\ \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} &= \frac{\left.\frac{\partial y}{\partial x}\right|_{x=B} - \left.\frac{\partial y}{\partial x}\right|_{x=A}}{\Delta x} \end{aligned}$$

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\end{aligned}$$

We need to take the limit of this expression as we consider an infinitesimal piece of string: $\Delta x \to 0$, $B \to A$.

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

where we use the definition of the second-order partial derivative.

The Wave Equation

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Remember that the speed of a wave on a string is

$$v = \sqrt{\frac{T}{\mu}}$$

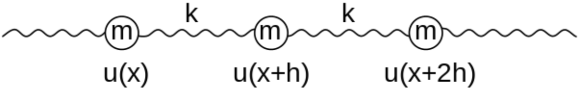
The wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Even though we derived this for a string, it applies much more generally!

Proof for Longitudinal Waves - Skipping

We can model longitudinal waves like sound waves by a series of masses connected by springs, length h.



u_1 $\quad$ u_2 $\quad$ u_3

u is a function that gives the displacement of the mass at each equilibrium position x, $x + h$, *etc.*

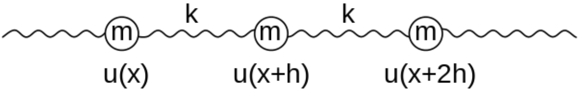
For such a case, the propagation speed is

$$v = \sqrt{\frac{KL}{\mu}}$$

where K is the spring constant of the entire spring chain, L is the length, and μ is the mass density.

[1]Figure from Wikipedia, by Sebastian Henckel.

The Wave Equation - Skipping



u_1 u_2 u_3

u is a function that gives the displacement of the mass at each equilibrium position x, $x+h$, *etc.*

Consider the mass, m, at equilibrium position $x+h$

$$
\begin{aligned}
F &= ma \\
k(u_3 - u_2) - k(u_2 - u_1) &= m\frac{\partial^2 u}{\partial t^2} \\
\frac{m}{k}\frac{\partial^2 u}{\partial t^2} &= u_3 - 2u_2 + u_1
\end{aligned}
$$

[1] Figure from Wikipedia, by Sebastian Henckel.

The Wave Equation - Skipping

$$\frac{m}{k}\frac{\partial^2 u}{\partial t^2} = u_3 - 2u_2 + u_1$$

We can re-write $\frac{m}{k}$ in terms of quantities for the entire spring chain. Suppose there are N masses.

$m = \frac{\mu L}{N}$ and $k = NK$ and $N = \frac{L}{h}$

$$\frac{\mu}{KL}\frac{\partial^2 u}{\partial t^2} = \frac{u(x+2h) - 2u(x+h) + u(x)}{h^2}$$

Letting $N \to \infty$ and $h \to 0$, the RHS is the definition of the 2nd derivative. Same equation!

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The Wave Equation

The wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

We derived this for a case of transverse waves (wave on a string) and a case of longitudinal waves (spring with mass).

It applies generally!

Solutions to the Wave Equation

Earlier we reasoned that a function of the form:

$$y(x,t) = f(x \pm vt)$$

should describe a propagating wave pulse.

Notice that f does not depend arbitrarily on x and t. It only depends on the two *together* by depending on $u = x \pm vt$.

Does it satisfy the wave equation?

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

(Next lecture...)

Summary

- wave speed on a string
- pulse propagation
- the wave equation

Homework Serway & Jewett (suggested, to try):

- Ch 16, onward from page 499. OQs: 5; Probs: 53, 60