Waves
Solutions to the Wave Equation
Sine Waves

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May 20, 2020

Last time

- pulse propagation
- the wave equation

Overview

- solutions to the wave equation
- sine waves

Solutions to the Wave Equation

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Does it satisfy the wave equation?

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Solutions to the Wave Equation

Does $y(x,t) = f(x - vt)$ satisfy the wave equation?

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and

$$\frac{\partial y}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial y}{\partial u} = -v\, f_u' \qquad ; \qquad \frac{\partial^2 y}{\partial t^2} = v^2 f_u''$$

where f_u' is the partial derivative of f wrt u.

Solutions to the Wave Equation

Replacing $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ in the wave equation:

$$\begin{aligned} f_u'' &= \frac{1}{v^2}(v^2) f_u'' \\ 1 &= 1 \end{aligned}$$

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In fact, any solution to the wave equation can be written:

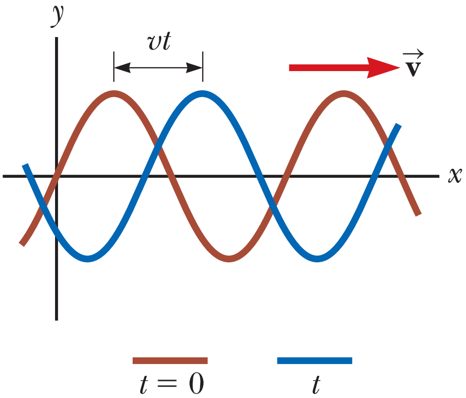
$$y(x, t) = f(x - vt) + g(x + vt)$$

Sine Waves

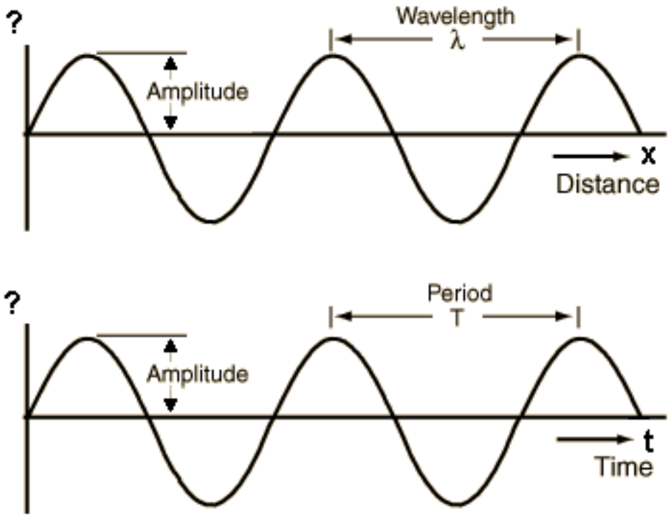
An important form of the function f is a sine or cosine wave. (All called "sine waves"). $y(x,t) = A\sin\big(B(x - vt) + C\big)$

This is the simplest periodic, continuous wave.

It is the wave that is formed by a (driven) simple harmonic oscillator connected to the medium.



Wave Quantities



Wave Quantities

wavelength, λ

the distance from one crest of the wave to the next, or the distance covered by one cycle.
units: length (m)

time period, T

the time for one complete oscillation.
units: time (s)

Sine Waves

frequency, f

the number of oscillations per second.

$$f = \frac{1}{T}$$

units: per time (Hz)

angular frequency, ω

the rate of change of phase of the wave.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

units: per time (rad/s)

Wave speed

How does wavelength relate to wave speed?

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

It travels the distance of one complete cycle in the time for one complete cycle.

$$v = \frac{\lambda}{T}$$

But since frequency is the inverse of the time period, we can relate speed to frequency and wavelength:

$$v = f\lambda$$

Wave speed

We also define a new quantity.

Wave number, k

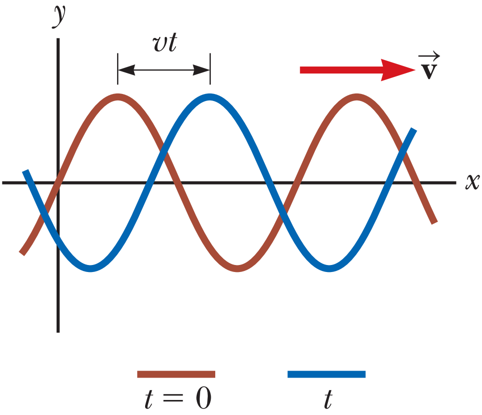
$$k = \frac{2\pi}{\lambda}$$

units: m^{-1}

Since $\omega = 2\pi f$ and $k = \frac{2\pi}{\lambda}$ this gives another way to express the speed of the wave:

$$v = \frac{\omega}{k}$$

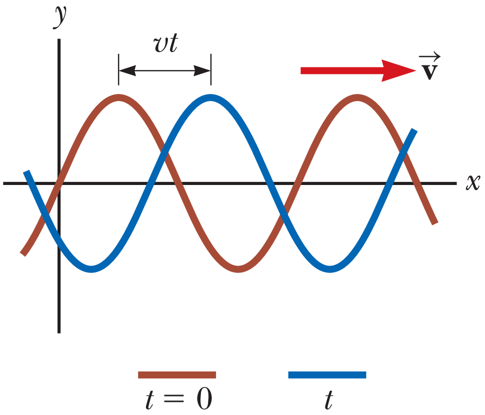
Sine Waves



$$y(x, t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt) + \phi\right)$$

This is usually written in a slightly different form...

Sine Waves



$$y(x,t) = A\sin(kx - \omega t + \phi)$$

where ϕ is a phase constant.

Summary

- solutions to the wave equation
- sine waves (covered in lab)

Homework Serway & Jewett (Could start looking at these):

- Ch 16, onward from page 499. OQs: 3, 9; CQs: 5; Probs: 5, 9, 11, 19, 41, 43