Waves
Reflection and Transmission
Standing Waves

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May 27, 2020

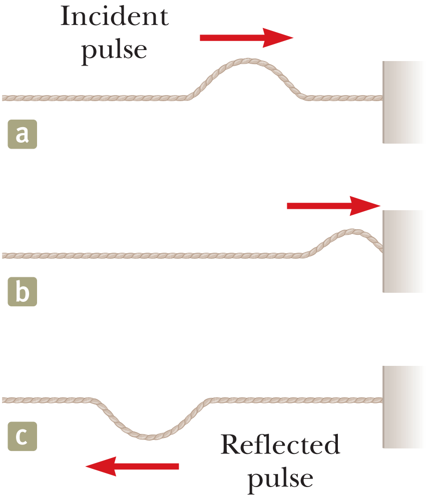
Last time

- interference of sine waves with same freq, different amplitudes
- boundary conditions
- reflection and transmission

Overview

- boundaries, reflection and transmission
- standing waves

Wave Reflection from a fixed end point



The reflected pulse is inverted. How does this happen?

Wave Reflection from a fixed end point

The boundary condition for a fixed end point at position $x = 0$ is:

$$y(x = 0, t) = 0$$

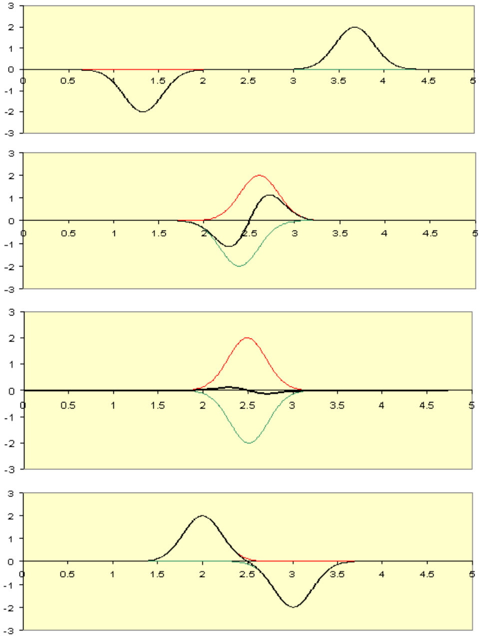
At any time, the point of the string at $x = 0$ cannot have any vertical displacement. It is tied to a wall!

The wave function for single pulse on the string does not satisfy this boundary condition.

$$y_1(x, t) = f(x - vt)$$

This pulse will continue in the $+x$ direction forever, past the end of the string. Makes no sense.

What if we imagine the string continues inside the wall, and there is a pulse traveling behind the wall in the $-x$ direction?



[1]Wall at $x = 2.5$. Digrams by Michal Fowler http://galileo.phys.virginia.edu

Wave Reflection from a fixed end point

If we allow another wave function:

$$y_2(x,t) = -f(-x - vt)$$

the total wave function will satisfy the boundary condition!

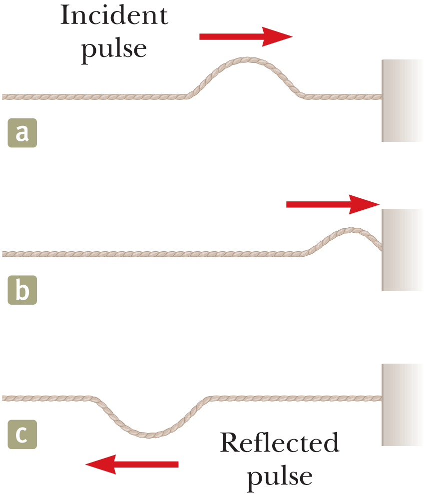
$$\begin{aligned} y(x,t) &= y_1(x,t) + y_2(x,t) \\ y(x,t) &= f(x - vt) + [-f(-x - vt)] \\ y(x=0,t) &= 0 \end{aligned}$$

$-f(-x - vt)$ corresponds to an inverted wave pulse that is also flipped left-to-right.

The reflected pulse is inverted.

Wave Reflection from a fixed end point

The reflected pulse is inverted.



Wave Reflection from a freely movable end point

Now we have a different boundary condition.

The *slope* of the string at the boundary must be zero.

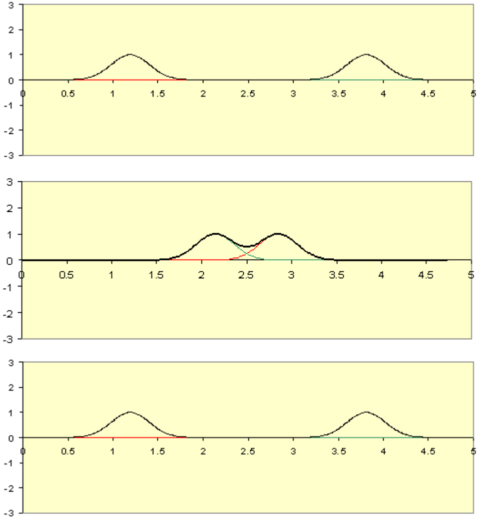
$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0$$

This ensures that the string will stay attached to the wall and the wall puts a horizontal force on the string.

To satisfy this boundary condition, imagine there is another pulse that is upright but moving in the $-x$ direction.

Wave Reflection from a freely movable end point

Imagine the free end of the string at $x = 2.5$. The slope there is zero at all times.



Wave Reflection from a freely movable end point

The new boundary condition is satisfied if $y_2 = f(-x - vt)$:

Let $u_1 = x - vt$ and $u_2 = -x - vt$.

$$\begin{aligned} y(x,t) &= f(x - vt) + f(-x - vt) \\ \frac{\partial y(x,t)}{\partial x} &= \frac{\partial f(u_1)}{\partial x} + \frac{\partial f(u_2)}{\partial x} \\ &= f'(u_1) + (-1) f'(u_2) \end{aligned}$$

The terms cancel when $u_1 = u_2$, that is, at $x = 0$.

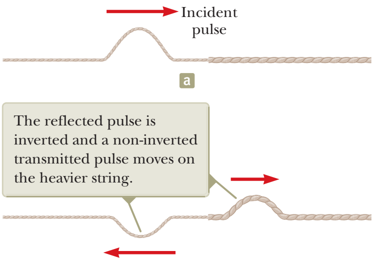
$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0$$

The pulse $f(-x - vt)$ is not inverted, but is reflected left-to-right.

Transmitted and Reflected Waves at a Boundary

If two ropes of different linear mass densities, μ_1 and μ_2 are attached together (under the same tension), an incoming pulse will be partially transmitted and partially reflected.

$\mu_1 < \mu_2$



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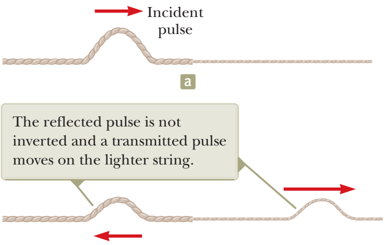


[1]Serway & Jewett, page 495.

Transmitted and Reflected Waves at a Boundary

The boundary conditions in this case are different again:

Now the y, the displacement, and $\frac{\partial y}{\partial x}$, slope of the string, must be continuous at the boundary.

Transmitted and Reflected Waves at a Boundary

Suppose $y_i(x,t) = f(x - v_1 t)$.

Then the reflected wave is

$$y_r = a_r\, f(-x - v_1 t)$$

and the transmitted wave is

$$y_t = a_t\, f\left(\frac{v_1}{v_2}(x - v_2 t)\right) .$$

a_r is the *reflection coefficient* (which is negative if the wave function is inverted) and a_t is the *transmission coefficient*.

To apply the boundary condition on the displacements:

$$\begin{aligned} y_i|_{x=0} + y_r|_{x=0} &= y_t|_{x=0} \\ f(-v_1 t) + a_r\, f(-v_1 t) &= a_t\, f(-v_1 t) \\ 1 + a_r &= a_t \end{aligned}$$

Wave Reflection from a freely movable end point

And using the boundary condition:

$$\left.\frac{\partial y_i(x,t)}{\partial x}\right|_{x=0} + \left.\frac{\partial y_r(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial y_t(x,t)}{\partial x}\right|_{x=0}$$

The reflection and transmission coefficients can be found:

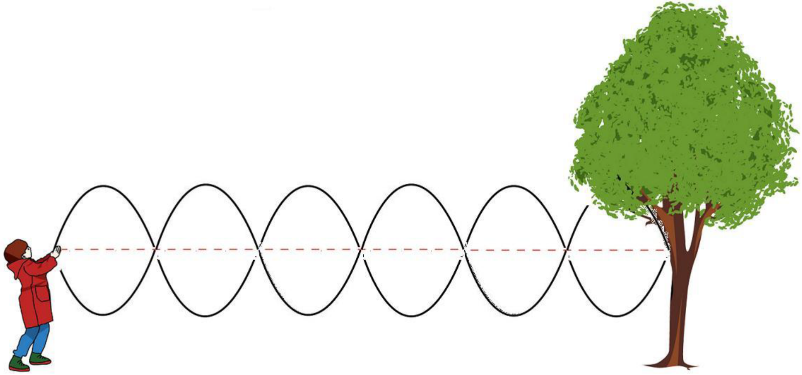
$$a_r = \frac{v_2 - v_1}{v_1 + v_2}$$

$$a_t = \frac{2v_2}{v_1 + v_2}$$

The height and width of the reflected and transmitted pulses are determined by the waves speeds (or equivalently, the string mass densities) on either side of the boundary.

Standing Waves

It is possible to create waves that do not seem to propagate.



They are produced by a wave moving to the left interfering with the wave reflected back the right.

Standing Waves

The incoming wave:

$$y_1(x, t) = A \sin(kx - \omega t)$$

Reflected wave:

$$y_2(x, t) = A \sin(kx + \omega t)$$

Using the trig identity:

$$\sin(\theta \pm \psi) = \sin\theta \cos\psi \pm \cos\theta \sin\psi$$

The resultant wave is:

$$y = [2A \sin(kx)] \ \cos(\omega t)$$

$$\uparrow \qquad\qquad \uparrow$$

Amplitude at x **SHM oscillation**

Standing Waves

$$y = [2A\sin(kx)]\ \cos(\omega t)$$

This does not correspond to a traveling wave!

It is a **standing wave**.

Points where $\sin kx = 0$ are called **nodes**. At these points the medium does not move.

Points where $\sin kx = \pm 1$ are called **antinodes**. At these points particles in the medium undergo their largest displacement.

Nodes and Antinodes

(Remember that $k = 2\pi/\lambda$)

Assuming $x = 0$ corresponds to a fixed point:

Nodes occur at

$$x = \frac{n\lambda}{2}$$

where n is an integer.

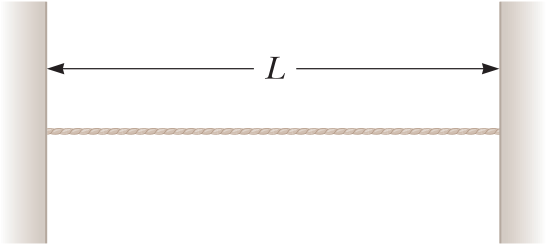
Antinodes occur at

$$x = \frac{(2n+1)\lambda}{4}$$

where again n is an integer.

Standing Waves and Resonance on a String

For a given string, fixed at both ends, only some wavelengths can correspond to standing waves.

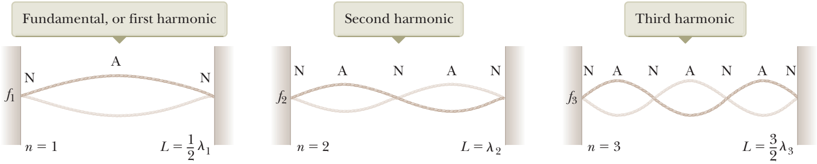


The boundary conditions are now

$$y(x = 0, t) = y(x = L, t) = 0$$

$x = 0$ and $x = L$ must be the positions of nodes.

Standing Waves and Resonance on a String



$\lambda_1 = 2L$ $\lambda_2 = L$ $\lambda_3 = \frac{2L}{3}$

Standing Waves and Resonance

These types of standing wave motions are called **normal modes**.

normal mode

A pattern of motion in a physical system where all parts of the system move sinusoidally with the same frequency and in phase.

Standing Waves and Resonance on a String

The wavelengths of these normal modes are given by the constraint $\sin(0) = \sin(kL) = 0$:

$$\lambda_n = \frac{2L}{n}$$

where n is a positive natural number (1, 2, 3...).

The frequencies that correspond to these wavelengths are called the **natural frequencies**:

$$f_n = \frac{nv}{2L} = n\, f_1$$

where n is a positive natural number.

For a string of density μ under tension T, the wave speed is constant $v = \sqrt{\frac{T}{\mu}}$.

Standing Waves and Resonance on a String

When a string is plucked, resonant (natural) frequencies tend to persist, while other waves at other frequencies are quickly dissipated.

Stringed instruments like guitars can be tuned by adjusting the tension in the strings.

While playing, pressing a string against a particular fret will change the string length or promote a specific harmonic.

Standing Waves and Resonance Question

Quick Quiz 18.3[1] When a standing wave is set up on a string fixed at both ends, which of the following statements is true?

(A) The number of nodes is equal to the number of antinodes.

(B) The wavelength is equal to the length of the string divided by an integer.

(C) The frequency is equal to the number of nodes times the fundamental frequency.

(D) The shape of the string at any instant shows a symmetry about the midpoint of the string.

[1] Serway & Jewett, page 543.

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Standing Waves and Resonance

In the following series of resonant frequencies, one frequency (lower than 400 Hz) is missing:

150, 225, 300, 375 Hz.

(a) What is the missing frequency?

(b) What is the frequency of the seventh harmonic?

Standing Waves and Resonance

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(a) 75 Hz (b) 525 Hz

Summary

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- standing waves