Inferential Statistics and Probability
a Holistic Approach

Chapter 7
Central Limit Theorem

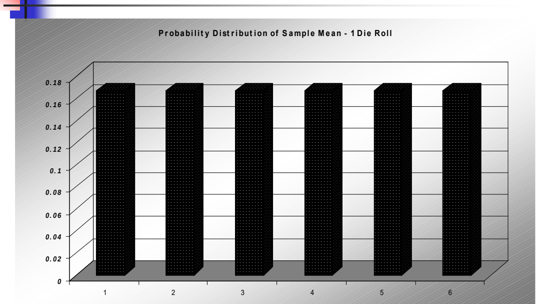
1

Distribution of Sample Mean

- Random Sample: $X_1, X_2, X_3, \ldots, X_n$
  - Each X_i is a Random Variable from the same population
  - All X_i's are Mutually Independent
- $\overline{X}$ is a function of Random Variables, so $\overline{X}$ is itself Random Variable.
- In other words, the Sample Mean can change if the values of the Random Sample change.
- What is the Probability Distribution of $\overline{X}$?

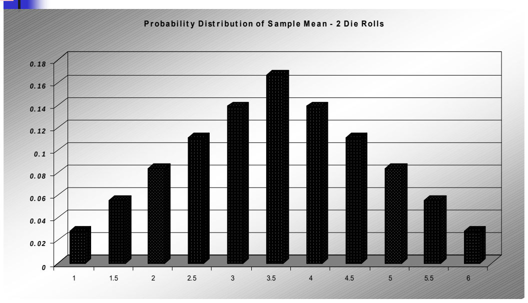
2

Example – Roll 1 Die



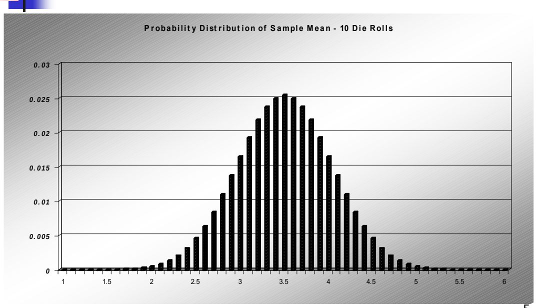
3

Example – Roll 2 Dice



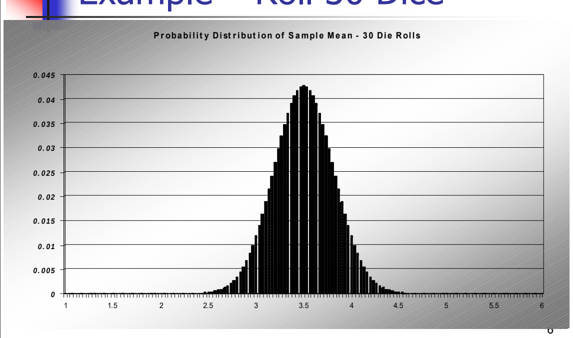
4

Example – Roll 10 Dice



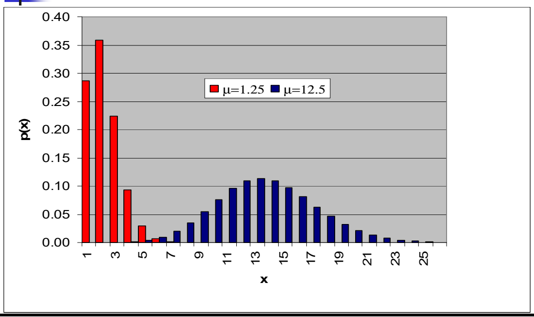
5

Example – Roll 30 Dice



6

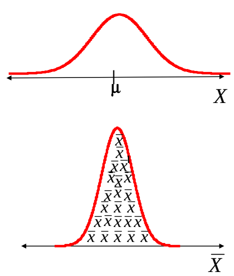
Example - Poisson



Central Limit Theorem – Part 1

- IF a Random Sample of **any size** is taken from a population with a **Normal Distribution** with mean= μ and standard deviation = σ
- THEN the distribution of the sample mean has a Normal Distribution with:

$\mu_{\bar{X}} = \mu \qquad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$

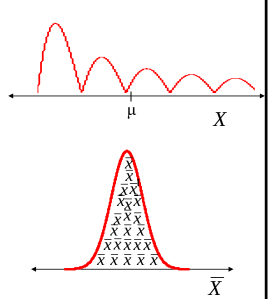


8

Central Limit Theorem – Part 2

- IF a random sample of **sufficiently large size** is taken from a population with **any Distribution** with mean= μ and standard deviation = σ
- THEN the distribution of the sample mean has approximately a Normal Distribution with:

$\mu_{\bar{X}} = \mu \qquad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$



9

Central Limit Theorem

3 important results for the distribution of $\overline{X}$

- Mean Stays the same

$$\mu_{\overline{X}} = \mu$$

- Standard Deviation Gets Smaller

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

- If n is sufficiently large, $\overline{X}$ has a Normal Distribution

10

Example

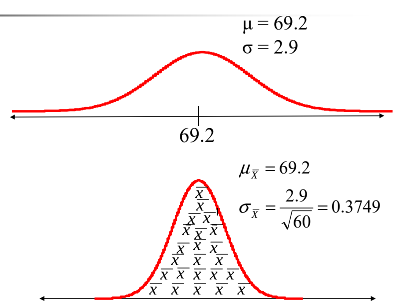
The mean height of American men (ages 20-29) is μ = 69.2 inches. If a random sample of 60 men in this age group is selected, what is the probability the mean height for the sample is greater than 70 inches? Assume σ = 2.9".

$$P(\overline{X} > 70) = P\left(Z > \frac{(70 - 69.2)}{2.9/\sqrt{60}} \right)$$

$$= P(Z > 2.14) = 0.0162$$

11

Example (cont)



12

Example – Central Limit Theorem

The waiting time until receiving a text message follows an exponential distribution with an expected waiting time of 1.5 minutes. Find the probability that the mean waiting time for the 50 text messages exceeds 1.6 minutes.

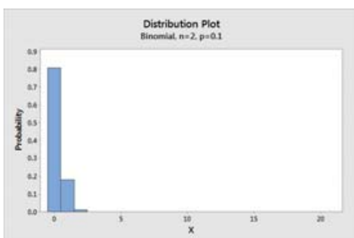
$\mu = 1.5 \qquad \sigma = 1.5 \qquad n = 50$

Use Normal Distribution (n>30)

$$P(\bar{X} > 1.6) = P\left(Z > \frac{(1.6 - 1.5)}{1.5/\sqrt{50}} \right) = P(Z > 0.47) = 0.3192$$

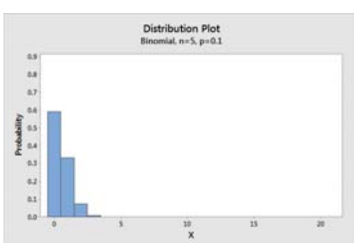
13

Binomial np=0.2



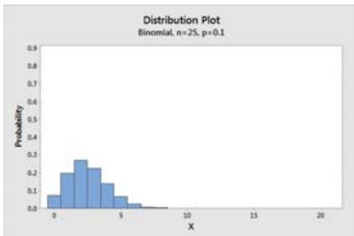
14

Binomial np=0.5



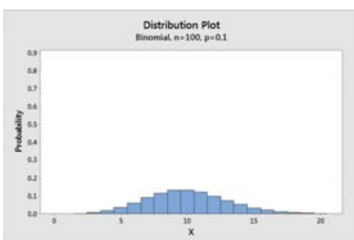
15

Binomial np=2.5



16

Binomial np=10



17

Central Limit Theorem
Sample Proportion

- The sample proportion of successes from a sample from a Binomial distribution is a random variable.
- If X is a random variable from a Binomial distribution with parameters n and p, an np > 10 and n(1-p) > 10, then the following is true for the Sample Proportion, $\hat{p}$:
- $\mu_{\hat{p}} = p \qquad \sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$
- The Distribution of $\hat{P}$ is approximately Normal.

18

Example

- 45% of all community college students in California receive fee waivers.
- Suppose you randomly sample 1000 community college students to determine the proportion of students with fee waivers in the sample.
- 483 of the sampled students are receiving fee waivers.
- Determine $\hat{P}$. Is the result unusual?

19