Inferential Statistics and Probability
a Holistic Approach

Chapter 13
Correlation and Linear Regression

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Mathematical Model

- You have a small business producing custom t-shirts.
- Without marketing, your business has revenue (sales) of $1000 per week.
- Every dollar you spend marketing will increase revenue by 2 dollars.
- Let variable X represent amount spent on marketing and let variable Y represent revenue per week.
- Write a mathematical model that relates X to Y

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Mathematical Model - Table

X=marketing	Y=revenue
$0	$1000
$500	$2000
$1000	$3000
$1500	$4000
$2000	$5000

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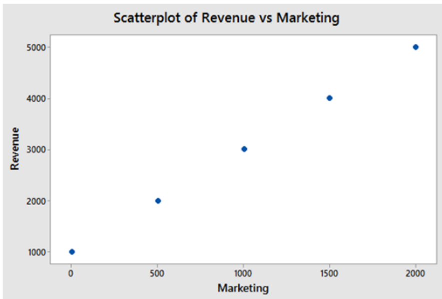
Mathematical Model - Scatterplot



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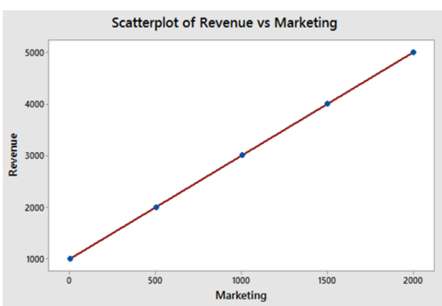
Mathematical Model - Linear



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Mathematical Linear Model

Linear Model	Example
$Y = \beta_0 + \beta_1 X$	$Y = 1000 + 2X$
Y: *Dependent Variable*	Y: *Revenue*
X: *Independent Variable*	X: *Marketing*
β_0: *Y-intercept*	β_0: \$1000
β_1: *Slope*	β_1: \$2 *per* \$1 *marketing*

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Statistical Model

- You have a small business producing custom t-shirts.
- Without marketing, your business has revenue (sales) of $1000 per week.
- Every dollar you spend marketing will increase revenue by an expected value of 2 dollars.
- Let variable X represent amount spent on marketing and let variable Y represent revenue per week.
- Let ε represent the difference between Expected Revenue and Actual Revenue (Residual Error)
- Write a statistical model that relates X to Y

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Statistical Model - Table

X=Marketing	Expected Revenue	Y=Actual Revenue	ε=Residual Error
$0	$1000	$1100	+$100
$500	$2000	$1500	-$500
$1000	$3000	$3500	+$500
$1500	$4000	$3900	-$100
$2000	$5000	$4900	-$100

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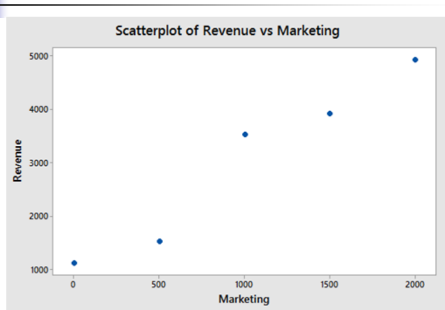
Statistical Model - Scatterplot



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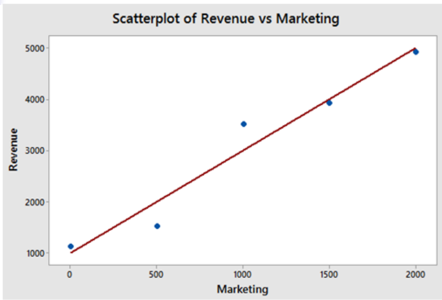
Statistical Model - Linear



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Statistical Linear Model

Regression Model

$Y = \beta_0 + \beta_1 X + \varepsilon$

Y: Dependent Variable

X: Independent Variable

β_0: Y-intercept

β_1: Slope

ε: $Normal(0, \sigma)$

Example

$Y = 1000 + 2X + \varepsilon$

Y: Revenue

X: Marketing

β_0: \$1000

β_1: \$2 per \$1 marketing

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Regression Analysis

- Purpose: to determine the regression equation; it is used to predict the value of the dependent response variable (Y) based on the independent explanatory variable (X).
- Procedure:
  - select a sample from the population
  - list the paired data for each observation
  - draw a scatter diagram to give a visual portrayal of the relationship
  - determine the regression equation.

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Simple Linear Regression Model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$
Y : *Dependent Variable*
X : *Independen t Variable*
β_0 : *Y – intercept*
β_1 : *Slope*
ε : *Normal* $(0, \sigma)$

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Estimation of Population Parameters

- From sample data, find statistics that will estimate the 3 population parameters
- Slope parameter
  - b_1 will be an estimator for β_1
- Y-intercept parameter
  - b_o 1 will be an estimator for β_o
- Standard deviation
  - s_e will be an estimator for σ

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Regression Analysis

- the regression equation: $\hat{Y} = b_0 + b_1 X$, where:
- $\hat{Y}$ is the average predicted value of *Y* for any *X*.
- b_0 is the Y-intercept, or the estimated *Y* value when *X=0*
- b_1 is the slope of the line, or the average change in $\hat{Y}$ for each change of one unit in *X*
- the least squares principle is used to obtain b_1 *and* b_0

$$SSX = \Sigma X^2 - \tfrac{1}{n}(\Sigma X)^2$$
$$SSY = \Sigma Y^2 - \tfrac{1}{n}(\Sigma Y)^2$$
$$SSXY = \Sigma XY - \tfrac{1}{n}(\Sigma X \cdot \Sigma y)$$

$$b_1 = \frac{SSXY}{SSX}$$
$$b_0 = \overline{Y} - b_1 \overline{X}$$

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Assumptions Underlying Linear Regression

- For each value of X, there is a group of Y values, and these Y values are *normally distributed*.
- The *means* of these normal distributions of Y values all lie on the straight line of regression.
- The *standard deviations* of these normal distributions are equal.
- The Y values are statistically independent. This means that in the selection of a sample, the Y values chosen for a particular X value do not depend on the Y values for any other X values.

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Example

- X = Average Annual Rainfall (Inches)
- Y = Average Sale of Sunglasses/1000
  - Make a Scatterplot
  - Find the least square line

X	10	15	20	30	40
Y	40	35	25	25	15

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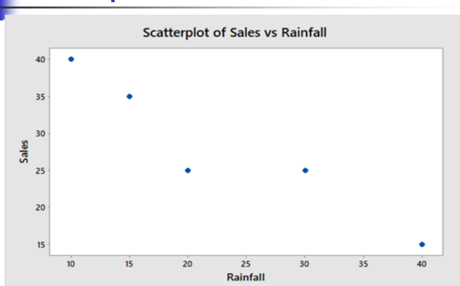
Example *continued*



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Example *continued*

	X	Y	X^2	Y^2	XY
	10	40	100	1600	400
	15	35	225	1225	525
	20	25	400	625	500
	30	25	900	625	750
	40	15	1600	225	600
Σ	115	140	3225	4300	2775

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Example *continued*

- Find the least square line
  - SSX = 580
  - SSY = 380
  - SSXY = -445
  - b_1 = -.767
  - b_0 = 45.647
  - $\hat{Y}$ = 45.647 - .767X

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The Standard Error of Estimate

- The standard error of estimate measures the scatter, or dispersion, of the observed values around the line of regression
- The formulas that are used to compute the standard error:

$$SSR = b_1 \cdot SSXY$$

$$SSE = \sum (Y - \hat{Y})^2 = SSY - SSR$$

$$MSE = SSE / (n-2)$$

$$s_e = \sqrt{MSE}$$

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Example *continued*

- Find SSE and the standard error:
  - SSR = 341.422
  - SSE = 38.578
  - MSE = 12.859
  - s_e = 3.586

x	y	ŷ	y - ŷ	$(y - ŷ)^2$
10	40	37.97	2.03	4.104
15	35	34.14	0.86	0.743
20	25	30.30	-5.30	28.108
30	25	22.63	2.37	5.620
40	15	14.96	0.04	0.002
			Total	38.578

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Correlation Analysis

- Correlation Analysis: A group of statistical techniques used to measure the strength of the relationship (correlation) between two variables.
- Scatter Diagram: A chart that portrays the relationship between the two variables of interest.
- Dependent Variable: The variable that is being predicted or estimated. "Effect"
- Independent Variable: The variable that provides the basis for estimation. It is the predictor variable. "Cause?" (Maybe!)

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The Coefficient of Correlation, r

- The Coefficient of Correlation (r) is a measure of the **strength** of the relationship between two variables.
  - It requires interval or ratio-scaled data (variables).
  - It can range from -1.00 to 1.00.
  - Values of -1.00 or 1.00 indicate perfect and strong correlation.
  - Values close to 0.0 indicate weak correlation.
  - Negative values indicate an inverse relationship and positive values indicate a direct relationship.

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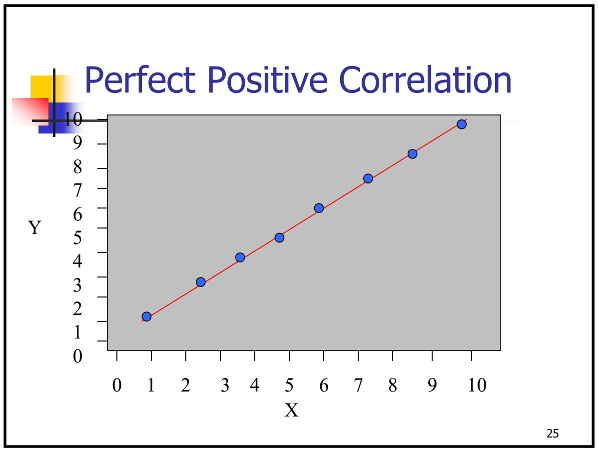
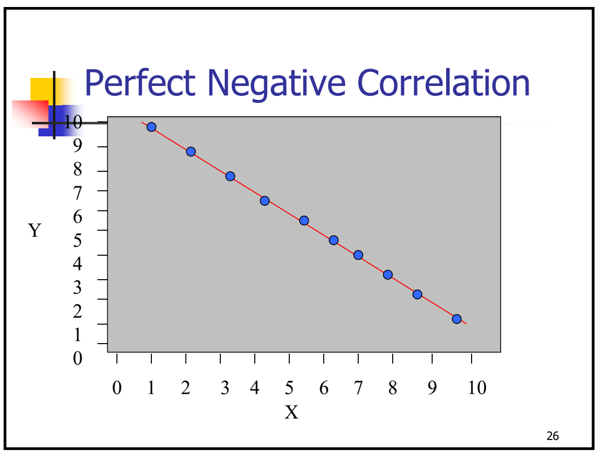
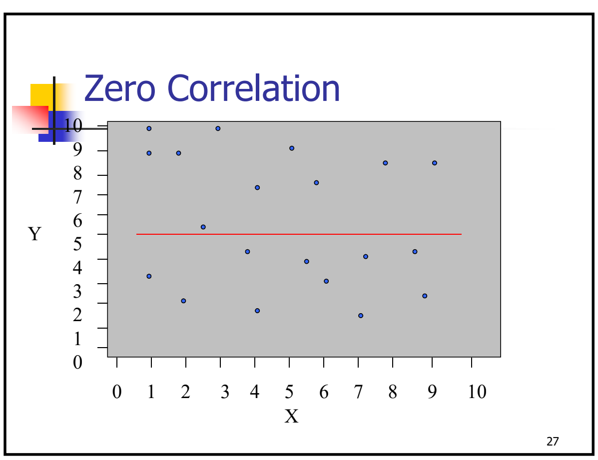

Perfect Positive Correlation
Y
10
9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10
X
25


Perfect Negative Correlation
Y
10
9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10
X
26


Zero Correlation
Y
10
9
8
7
6
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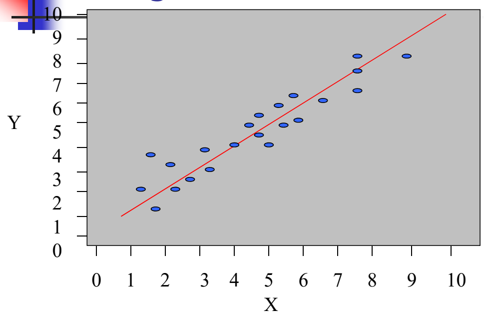
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10
X
27

Strong Positive Correlation



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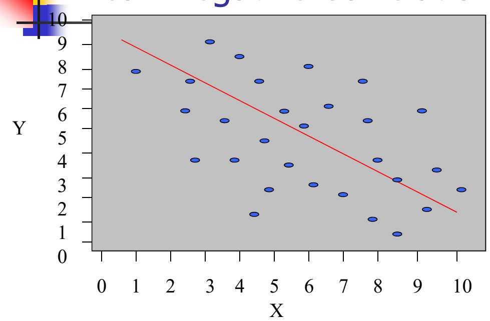
Weak Negative Correlation



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Causation

- Correlation does not necessarily imply causation.
- There are 4 possibilities if X and Y are correlated:
  1. X causes Y
  2. Y causes X
  3. X and Y are caused by something else.
  4. Confounding - The effect of X and Y are hopelessly mixed up with other variables.

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Causation - Examples

- City with more police per capita have more crime per capita.
- As Ice cream sales go up, shark attacks go up.
- People with a cold who take a cough medicine feel better after some rest.

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r^2: Coefficient of Determination

- r^2 is the proportion of the total variation in the dependent variable Y that is explained or accounted for by the variation in the independent variable X.
- The coefficient of determination is the square of the coefficient of correlation, and ranges from 0 to 1.

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Formulas for r and r^2

$$r = \frac{SSXY}{\sqrt{SSX \cdot SSY}} \qquad r^2 = \frac{SSR}{SSY}$$

$$SSX = \Sigma X^2 - \tfrac{1}{n}(\Sigma X)^2$$

$$SSY = \Sigma Y^2 - \tfrac{1}{n}(\Sigma Y)^2$$

$$SSXY = \Sigma XY - \tfrac{1}{n}(\Sigma X \cdot \Sigma Y)$$

$$SSR = SSY - \left(SSXY^2 / SSX\right)$$

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Example

- X = Average Annual Rainfall (Inches)
- Y = Average Sale of Sunglasses/1000

X	10	15	20	30	40
Y	40	35	25	25	15

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Example *continued*

- Make a Scatter Diagram
- Find r and r^2

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Example *continued*

X	Y	X^2	Y^2	XY
10	40	100	1600	400
15	35	225	1225	525
20	25	400	625	500
30	25	900	625	750
40	15	1600	225	600
115	140	3225	4300	2775

- SSX = $3225 - 115^2/5$ = 580
- SSY = $4300 - 140^2/5$ = 380
- SSXY= 2775 - (115)(140)/5 = -445

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Example *continued*

- r = -445/sqrt(580 x 330) = -.9479
  - Strong negative correlation
- r^2 = .8985
  - About 89.85% of the variability of sales is explained by rainfall.

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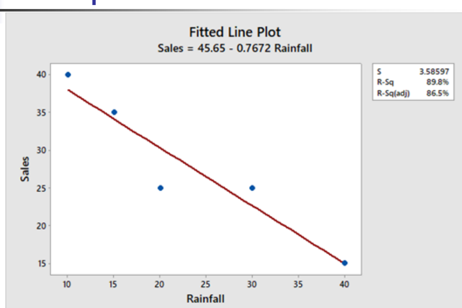
Example *continued*



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Characteristics of F-Distribution

- There is a "family" of *F* Distributions.
- Each member of the family is determined by two parameters: the numerator degrees of freedom and the denominator degrees of freedom.
- *F* cannot be negative, and it is a continuous distribution.
- The *F* distribution is positively skewed.
- Its values range from 0 *to* ∞. As $F \rightarrow \infty$ the curve approaches the X-axis.

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Hypothesis Testing in Simple Linear Regression

- The following Tests are equivalent:
  - H_0: X and Y are uncorrelated
  - H_a: X and Y are correlated
  - H_0: $\beta_1 = 0$
  - H_a: $\beta_1 \neq 0$
- Both can be tested using ANOVA

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ANOVA Table for Simple Linear Regression

Source	SS	df	MS	F
Regression	SSR	1	SSR/dfR	MSR/MSE
Error/Residual	SSE	n-2	SSE/dfE	
TOTAL	SSY	n-1		

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Example *continued*

- Test the Hypothesis $H_o: \beta_1 = 0$, α=5%

Source	SS	df	MS	F	p-value
Regression	341.422	1	341.422	26.551	0.0142
Error	38.578	3	12.859		
TOTAL	380.000	4			

- Reject Ho p-value < α

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Confidence Interval

- The confidence interval for the mean value of Y for a given value of X is given by:

$$\hat{Y} \pm t \cdot s_e \cdot \sqrt{\frac{1}{n} + \frac{(X - \overline{X})^2}{SSX}}$$

- Degrees of freedom for t =n-2

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Prediction Interval

- The prediction interval for an individual value of Y for a given value of X is given by:

$$\hat{Y} \pm t \cdot s_e \cdot \sqrt{1 + \frac{1}{n} + \frac{(X - \overline{X})^2}{SSX}}$$

- Degrees of freedom for t =n-2

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Example *continued*

- Find a 95% Confidence Interval for Sales of Sunglasses when rainfall = 25 inches.
- Find a 95% Prediction Interval for Sales of Sunglasses when rainfall = 25 inches.

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Example – Minitab output

- Sales = 45.65 - 0.767 Rainfall
- Variable Setting
- Rainfall   25
- Fit   SE Fit   95% CI   95% PI
- 26.4655 1.63111 (21.2746, 31.6564) (13.9282, 39.0028)

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Example *continued*

- 95% Confidence Interval

$$22.63 \pm 6.60$$

- 95% Confidence Interval

$$22.63 \pm 13.18$$

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Residual Analysis

- Residuals should
  - have a normal distribution with constant σ
  - be mutually independent
  - not follow a pattern
  - be checked for outliers
    - with respect the line
    - with respect to X

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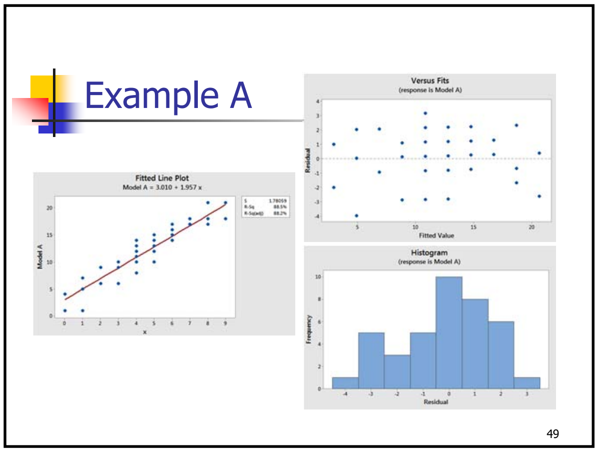
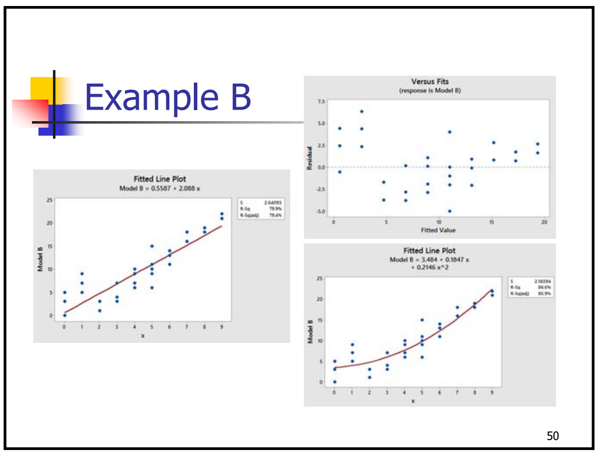
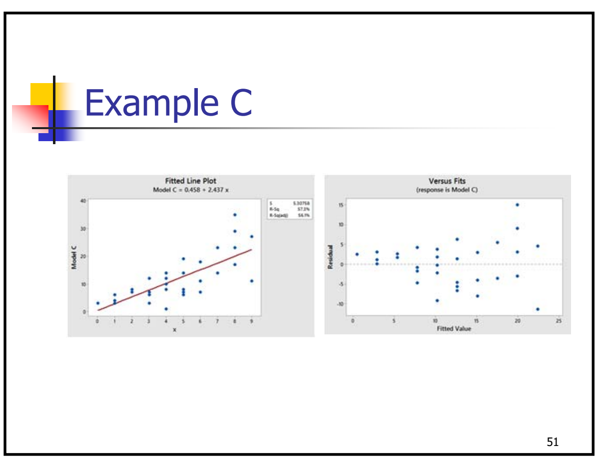

Example A
Versus Fits
(response is Model A)
Residual
Fitted Value
Fitted Line Plot
Model A = 3.010 + 1.957 x
S 1.78059
R-Sq 88.5%
R-Sq(adj) 88.2%
Model A
x
Histogram
(response is Model A)
Frequency
Residual
49


Example B
Versus Fits
(response is Model B)
Residual
Fitted Value
Fitted Line Plot
Model B = 0.5587 + 2.088 x
S 2.64093
R-Sq 79.9%
R-Sq(adj) 79.4%
Model B
x
Fitted Line Plot
Model B = 3.484 + 0.1847 x
+ 0.2146 x^2
S 2.18394
R-Sq 86.6%
R-Sq(adj) 85.9%
Model B
x
50


Example C
Fitted Line Plot
Model C = 0.458 + 2.437 x
S 5.30758
R-Sq 57.3%
R-Sq(adj) 56.1%
Model C
x
Versus Fits
(response is Model C)
Residual
Fitted Value
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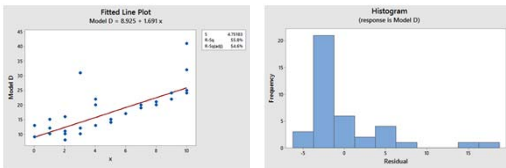
Example D



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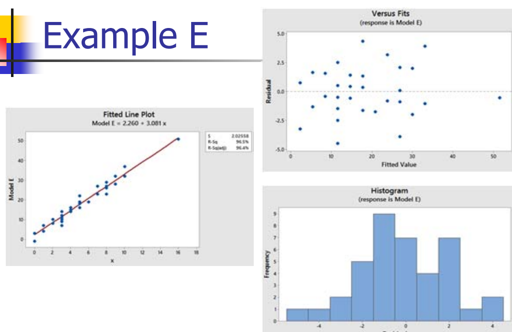
Example E



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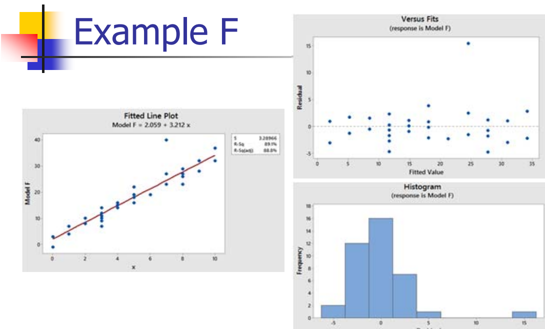
Example F



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Using Minitab to Run Regression

- Data shown is engine size in cubic inches (X) and MPG (Y) for 20 cars.

x	y		x	y
400	15		104	25
455	14		121	26
113	24		199	21
198	22		360	10
199	18		307	10
200	21		318	11
97	27		400	9
97	26		97	27
110	25		140	28
107	24		400	15

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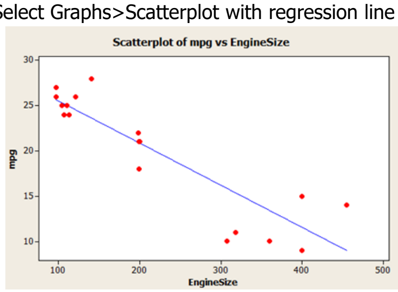
Using Minitab to Run Regression

Select Graphs>Scatterplot with regression line



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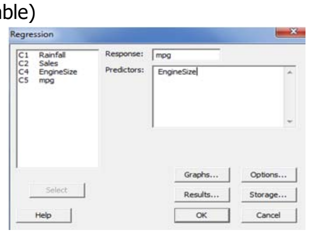
Using Minitab to Run Regression

Select Statistics>Regression>Regression, then choose the Response (Y-variable) and model (X-variable)



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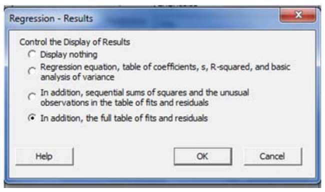
Using Minitab to Run Regression

Click the results box, and choose the fits and residuals to get all predictions.



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Using Minitab to Run Regression

The results at the beginning are the regression equation, the intercept and slope, the standard error of the residuals, and the r^2

```
The regression equation is
mpg = 30.2 - 0.0466 EngineSize
```

Predictor	Coef	SE Coef	T	P
Constant	30.203	1.361	22.20	0.000
EngineSize	-0.046598	0.005378	-8.66	0.000

```
S = 2.95688   R-Sq = 80.7%   R-Sq(adj) = 79.6%
```

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Using Minitab to Run Regression

Next is the ANOVA table, which tests the significance of the regression model.

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	656.42	656.42	75.08	0.000
Residual Error	18	157.38	8.74		
Total	19	813.80			

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Using Minitab to Run Regression

Finally, the residuals show the potential outliers.

Obs	EngineSize	mpg	Fit	SE Fit	Residual	St Resid
1	400	15.000	11.564	1.167	3.436	1.26
2	455	14.000	9.001	1.421	4.999	1.93
3	113	24.000	24.937	0.880	-0.937	-0.33
4	198	22.000	20.976	0.673	1.024	0.36
5	199	18.000	20.930	0.672	-2.930	-1.02
6	200	21.000	20.883	0.671	0.117	0.04
7	97	27.000	25.683	0.939	1.317	0.47
8	97	26.000	25.683	0.939	0.317	0.11
9	110	25.000	25.077	0.891	-0.077	-0.03
10	107	24.000	25.217	0.902	-1.217	-0.43
11	104	25.000	25.357	0.913	-0.357	-0.13
12	121	26.000	24.565	0.853	1.435	0.51
13	199	21.000	20.930	0.672	0.070	0.02
14	360	10.000	13.427	0.998	-3.427	-1.23
15	307	10.000	15.897	0.807	-5.897	-2.07R
16	318	11.000	15.385	0.842	-4.385	-1.55
17	400	9.000	11.564	1.167	-2.564	-0.94
18	97	27.000	25.683	0.939	1.317	0.47
19	140	28.000	23.679	0.792	4.321	1.52
20	400	15.000	11.564	1.167	3.436	1.26

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Using Minitab to Run Regression

- Find a 95% confidence interval for the **expected** MPG of a car with an engine size of 250 ci.
- Find a 95% prediction interval for the **actual** MPG of a car with an engine size of 250 ci.

```
mpg = 30.20 - 0.04660 EngineSize


Variable    Setting
EngineSize      250


    Fit    SE Fit        95% CI              95% PI
18.5533  0.679201  (17.1264, 19.9803)  (12.1793, 24.9273)
```

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Residual Analysis

- Residuals for Simple Linear Regression
  - The residuals should represent a linear model.
  - The standard error (standard deviation of the residuals) should not change when the value of X changes.
  - The residuals should follow a normal distribution.
  - Look for any potential extreme values of X.
  - Look for any extreme residual errors

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