Continuous Distributions

- "Uncountable" Number of possibilities
- Probability of a point makes no sense
- Probability is measured over intervals
- Comparable to Relative Frequency
  Histogram – Find Area under curve.

Continuous Random Variable

- $f(x)$ is a density function
- $P(X<x)$ is a distribution function.
- $P(a<X<b) =$ area under function between $a$ and $b$

Examples of Exponential Distribution

- Time until...
  - a circuit will fail
  - the next RM 7 Earthquake
  - the next customer calls
  - An oil refinery accident
  - you buy a winning lotto ticket

Exponential distribution

- Waiting time
- "Memoryless"
- $f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$
- $P(x>a) = e^{-\frac{a}{\mu}}$
- $\mu = \mu, \sigma^2 = \mu^2$
- $P(x>a+b|X>b) = e^{-\frac{a}{\mu}}$
Relationship between Poisson and Exponential Distributions

- If occurrences follow a Poisson Process with mean $= \mu$, then the waiting time for the next occurrence has Exponential distribution with mean $= 1/\mu$.

- Example: If accidents occur at a plant at a constant rate of 3 per month, then the expected waiting time for the next accident is $1/3$ month.

Exponential Example

The life of a digital display of a calculator has exponential distribution with $\mu=500$ hours.

(a) Find the chance the display will last at least 600 hours.

$$P(x>600) = e^{-600/500} = e^{-1.2} = 0.3012$$

(b) Assuming it has already lasted 500 hours, find the chance the display will last an additional 600 hours.

$$P(x>1100|x>500) = P(x>600) = 0.3012$$

Uniform Distribution

- Rectangular distribution
- Example: Random number generator

$$f(x) = \frac{1}{b-a} \quad a \leq x \leq b$$

$$\mu = E(X) = \frac{b+a}{2}$$

$$\sigma^2 = Var(X) = \frac{(b-a)^2}{12}$$

Uniform Distribution - Probability

$$P(c < X < d) = \frac{d-c}{b-a}$$

Uniform Distribution - Percentile

Formula to find the $p$th percentile $X_p$:

$$X_p = a + p(b-a)$$
Uniform Example 1

- Find mean, variance, $P(X<3)$ and 70th percentile for a uniform distribution from 1 to 11.

$$\mu = \frac{1+11}{2} = 6 \quad \sigma^2 = \frac{(11-1)^2}{12} = 8.33$$

$$P(X < 3) = \frac{3-1}{11-1} = 0.3$$

$$X_{70} = 1 + 0.7(11-1) = 8$$

Uniform Example 2

- A tea lover orders 1000 grams of Tie Guan Yin loose leaf when his supply gets to 50 grams.
- The amount of tea currently in stock follows a uniform random variable.
- Determine this model
- Find the mean and variance
- Find the probability of at least 700 grams in stock.
- Find the 80th percentile

Uniform Example 3

- A bus arrives at a stop every 20 minutes.
- Find the probability of waiting more than 15 minutes for the bus after arriving randomly at the bus stop.
- If you have already waited 5 minutes, find the probability of waiting an additional 10 minutes or more. (Hint: recalculate parameters a and b)

Normal Distribution

- The normal curve is bell-shaped
- The mean, median, and mode of the distribution are equal and located at the peak.
- The normal distribution is symmetrical about its mean. Half the area under the curve is above the peak, and the other half is below it.
- The normal probability distribution is asymptotic - the curve gets closer and closer to the x-axis but never actually touches it.

The Standard Normal Probability Distribution

- A normal distribution with a mean of 0 and a standard deviation of 1 is called the standard normal distribution.
- Z value: The distance between a selected value, designated $x$, and the population mean $\mu$, divided by the population standard deviation, $\sigma$

$$Z = \frac{X - \mu}{\sigma}$$

Areas Under the Normal Curve – Empirical Rule

- About 68 percent of the area under the normal curve is within one standard deviation of the mean. $\mu \pm \sigma$
- About 95 percent is within two standard deviations of the mean. $\mu \pm 2\sigma$
- 99.7 percent is within three standard deviations of the mean. $\mu \pm 3\sigma$
EXAMPLE

The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.

- About 68% of the daily water usage per person in New Providence lies between what two values?
- \( \mu \pm 1\sigma = 20 \pm 1(5) \). That is, about 68% of the daily water usage will lie between 15 and 25 gallons.

Normal Distribution – probability problem procedure

- Given: Interval in terms of \( X \)
- Convert to \( Z \) by \( Z = \frac{X - \mu}{\sigma} \)
- Look up probability in table.

EXAMPLE

The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.

- What is the probability that a person from the town selected at random will use less than 18 gallons per day?
- The associated \( Z \) value is \( Z = \frac{18 - 20}{5} = -0.40 \).
- Thus, \( P(X < 18) = P(Z < -0.40) = 0.3446 \)

EXAMPLE continued

- What proportion of the people uses between 18 and 24 gallons?
- The \( Z \) value associated with \( X = 18 \) is \( Z = -0.40 \) and with \( X = 24 \), \( Z = \frac{24 - 20}{5} = 0.80 \).
- Thus, \( P(18 < X < 24) = P(-0.40 < Z < 0.80) = 0.7881 - 0.3446 = 0.4435 \)

EXAMPLE continued

- What percentage of the population uses more than 26.2 gallons?
- The \( Z \) value associated with \( X = 26.2 \), \( Z = \frac{26.2 - 20}{5} = 1.24 \).
- Thus \( P(X > 26.2) = P(Z > 1.24) = 1 - 0.8925 = 0.1075 \)

Normal Distribution – percentile problem procedure

- Given: probability or percentile desired.
- Look up \( Z \) value in table that corresponds to probability.
- Convert to \( X \) by the formula:

\[
X = \mu + Z\sigma
\]
EXAMPLE

The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons. A special tax is going to be charged on the top 5% of water users.

- Find the value of daily water usage that generates the special tax
- The Z value associated with 95th percentile =1.645
- $X = 20 + 5(1.645) = 28.2$ gallons per day

EXAMPLE

Professor Kurv has determined that the final averages in his statistics course is normally distributed with a mean of 77.1 and a standard deviation of 11.2.

- He decides to assign his grades for his current course such that the top 15% of the students receive an A.
- What is the lowest average a student can receive to earn an A?
  - The top 15% would be the finding the 85th percentile. Find $k$ such that $P(X<k)=.85$.
  - The corresponding Z value is 1.04. Thus we have $X = 77.1 + (1.04)(11.2)$, or $X = 88.75$

EXAMPLE

The amount of tip the servers in an exclusive restaurant receive per shift is normally distributed with a mean of $80 and a standard deviation of $10.

- Shelli feels she has provided poor service if her total tip for the shift is less than $65.
- What percentage of the time will she feel like she provided poor service?
  - Let $y$ be the amount of tip. The Z value associated with $X=65$ is $Z=(65-80)/10=-1.5$.
  - Thus $P(X<65)=P(Z<-1.5)=.0668$.

Distribution of Sample Mean

- Random Sample: $X_1, X_2, X_3, \ldots, X_n$
  - Each $X_i$ is a Random Variable from the same population
  - All $X_i$s are Mutually Independent
- $\bar{X}$ is a function of Random Variables, so $\bar{X}$ is itself Random Variable.
  - In other words, the Sample Mean can change if the values of the Random Sample change.
- What is the Probability Distribution of $\bar{X}$?
Example – Roll 10 Dice

Example – Roll 30 Dice

Example - Poisson

Central Limit Theorem – Part 1

Central Limit Theorem – Part 2

Central Limit Theorem

3 important results for the distribution of $\bar{X}$

- Mean Stays the same
  $\mu_{\bar{X}} = \mu$

- Standard Deviation Gets Smaller
  $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$

- If $n$ is sufficiently large, $\bar{X}$ has a Normal Distribution
Example

The mean height of American men (ages 20-29) is \( \mu = 69.2 \) inches. If a random sample of 60 men in this age group is selected, what is the probability the mean height for the sample is greater than 70 inches? Assume \( \sigma = 2.9 \) inches.

\[
P(\bar{X} > 70) = P\left( Z > \frac{70 - 69.2}{2.9/\sqrt{60}} \right)
\]

\[= P(Z > 2.14) = 0.0162\]

Example (cont)

Example – Central Limit Theorem

The waiting time until receiving a text message follows an exponential distribution with an expected waiting time of 1.5 minutes. Find the probability that the mean waiting time for the 50 text messages exceeds 1.6 minutes.

\[
\mu = 1.5 \quad \sigma = 1.5 \quad n = 40
\]

Use Normal Distribution (n>30)

\[
P(\bar{X} > 1.6) = P\left( Z > \frac{1.6 - 1.5}{1.5/\sqrt{50}} \right) = P(Z > 0.47) = 0.3192
\]