Lesson 4.2.2
Exponential Models & Logarithms

SACWAY
BRAINSTORMING ALGEBRA & STATISTICS

STUDENT NAME ______________________ DATE ____________

INTRODUCTION

Compound Interest

When you invest money in a fixed-rate interest earning account, you receive interest at regular time intervals. The first time you receive interest, you receive a percentage of your initial investment (the *principal*). After that, you continue to receive interest from your principal, but your previous interest *also* earns interest. This excellent phenomenon is known as *compound interest*.

Suppose that you have a fixed-rate interest earning account which earns 8% interest at the end of every year, and that you have deposited \$10,000. Let's call the time of the initial investment $t = 0$.

1. How much money will be in the account after one year when $t = 1$? (Remember to count principal *plus* interest!)

2. What percent of the original principal will we earn as interest?

3. What percent of the original principal do we get to keep?

4. Adding these percentages together, what is the total percentage of the principal that becomes the new balance?

5. How does this percentage appear in decimal form, if we wish to use it for computing account balances?

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The value you determined in step 5 is a multiplier for determining a given year's balance. If we multiply the principal by this value 3 times, we would get the balance on the third year. Fill in the balances for the given years in the table below. The year 2 balance is provided. See if you can get the same value.

t **(Years)**	***A*** **(Account Balance)**
0	10000
1	
2	11664
3	
4	
5	

6. Maybe you determined the year 4 balance by multiplying by the interest multiplier four times. How could you do this more efficiently by using the exponent button on your calculator?

7. Describe a method for determining the account balance in 25 years, using mathematical exponents.

8. Compute the balance for the years in the table below. Use the exponent function on your calculator.

t **(Years)**	***A*** **(Account Balance)**
15	
25	

9. Write an equation that gives the account balance, A, after t years. The equation should use exponents.

$A =$

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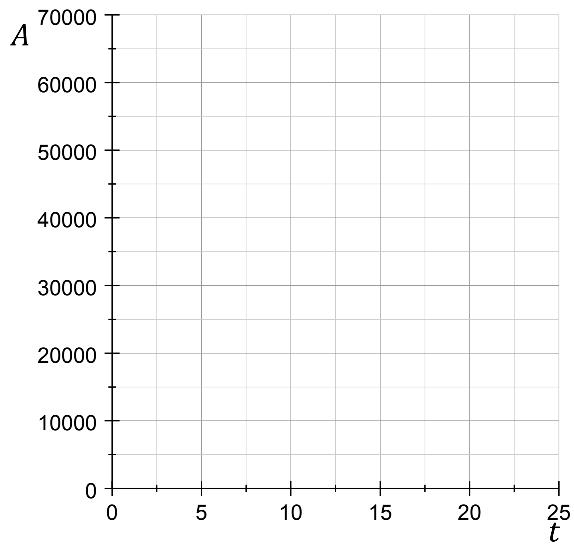
10. Graph your exponential function below. Note that you have already computed the balance for several years above. Compute a few additional points to help complete the picture.



11. Use your graph to estimate the time when the account will contain $40000.

NEXT STEPS

If all went as planned, you discovered a formula for computing compound interest, when interest is earned once each year. If P is the principal, the initial deposit, of some investment into a fixed-rate interest earning account, and r is the annual interest rate, then the multiplier for determining the next year's balance is $1 + r$ (keep 100% of the original investment and add in a proportion of r for interest. Remember, the multiplier needs to be in decimal, not percentage, form). The balance at year $t = 1$ is $A = P(1 + r)$. The balance at year t = 2 is
$A = P(1 + r)^2$.

At year t, the balance is

$$A = P(1 + r)^t$$

This is an *exponential* function, because the only changing quantity is the variable t in the *exponent*. The multiplier, 1 + r, is the *base* of the exponent.

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TRY THESE

12. Suppose you invest \$5000 at 12% annual interest in a fixed-rate interest earning account, where interest is compounded once each year. What will the account balance be in 10 years?

13. What will the account balance be in 30 years?

NEXT STEPS

In reality, interest earning accounts receive interest more than once a year. Suppose an account earns 8% interest, compounded quarterly (four times a year). When this occurs, the 8% is divided by four, and you only receive 2% at each compounding. This is actually better because it gives your interest more times to compound, and in the end you earn slightly more than 8% annually. If the investment is for 10 years, you would actually receive interest $4 \cdot 10 = 40$ times.

In general, when the annual interest rate is r, and interest is compounded *n* times each year, the actual rate is r/n, so the multiplier for computing interest is $1 + r/n$. Interest is earned $n \cdot t$ times, so the exponent for the multiplier is $n \cdot t$. These ideas give a new compound interest formula,

$$A = P(1 + r/n)^{nt}.$$

In this equation, *A* is account balance, *P* is principal, *r* is annual interest rate, *t* is time in years, and *n* is the number of times interest is compounded each year.

TRY THESE

14. Suppose you invest \$2500 at 10%, compounded quarterly, in a fixed-rate interest earning account. How much will you have in 17 years?

15. How much will you have in 25 years?

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NEXT STEPS

In our last lesson, we learned how to use logarithms to solve for an exponent in an equation. We learned that

$$\log_b(x) = \text{the power to which } b \text{ must be raised to yield } x,$$

and that, to evaluate $\log_b(x)$, we simply ask "b to the *what* yields x?". For example, to evaluate $\log_6(216)$, we ask, "6 raised to the *what* yields 216?". Since $6^3 = 216$, the answer is 3.

$$\log_6(216) = 3$$

TRY THESE

Evaluate the following logarithms *without* a calculator.

16. $\log_2(64) =$

17. $\log_5(125) =$

18. $\log_9(81) =$

19. $\log_4(4^7) =$

20. $\log_6(6^{15}) =$

21. $\log_b(b^x) =$

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NEXT STEPS

Also in our last lesson, we learned to evaluate logarithms with a calculator. We learned that calculators have a special logarithm function called the *natural logarithm*, which is denoted as *ln*. With the natural logarithm, we can evaluate any logarithm as follows.

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}.$$

TRY THESE

Evaluate the following logarithms *with* a calculator.

22. $\log_6(100)$ =

23. $\log_8(475)$ =

24. $\log_{10.5}(1298)$ =

25. $\log_7(7^{123})$ =

26. $\log_{12}(12^{26})$ =

NEXT STEPS

If you paid attention during the two previous problem sets, you may have discovered a principal in logarithms. If you did not see it, check again. Look at problems 19, 20, 24, 25, 26, but look particularly at problem 21, which says it all. If this was done correctly it tells us that

$$\log_b(b^u) = u.$$

This key idea is really the main purpose of logarithms. It tells us that if a logarithm is applied to an exponential of the same base, it gives us the exponent. The logarithm is the *inverse operation* of the exponential.

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Watch how we use this idea to solve the following equation.

$$1000 \cdot 1.10^{2x} = 20000$$

$1000 \cdot 1.10^{2x} = 20000$	Here is our given equation.
$\dfrac{1000 \cdot 1.10^{2x}}{1000} = \dfrac{20000}{1000}$	We need to isolate the exponential, so we divide by 1000.
$1.10^{2x} = 20$	The exponential is isolated. Its base is 1.10.
$\log_{1.1}(1.10^{2x}) = \log_{1.1}(20)$	Here we apply a logarithm of the same base to each side of the equation.
$2x = \log_{1.1}(20)$	Now we use the rule that $\log_b(b^u) = u$.
$2x = \dfrac{\ln(20)}{\ln(1.1)}$	Convert to a natural logarithm to evaluate on a calculator.
$2x \approx 31.4314$	We have isolated $2x$, but now we must divide by 2 to finish.
$\dfrac{2x}{2} \approx \dfrac{31.4314}{2}$	Divide by 2.
$x \approx 15.7157$	Here is the final answer.

We can verify this final answer by substituting into the original equation:

$$1000 \cdot 1.10^{31.4314} = 20000.002$$

The solution found above is quite close to being exact.

In summary, to solve an exponential equation,

I. Isolate the exponential (the base which is raised to a power).
II. Apply a logarithm of the same base as the exponential to each side.
III. Use the rule $\log_b(b^u) = u$ to simplify the equation.
IV. Evaluate the resulting expression with a calculator using the rule $\log_b(x) = \ln(x)/\ln(b)$.

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TRY THESE

Use the steps above to solve the equations below.

27. $3 \cdot 1.2^{3x} = 93$

28. $36 \cdot 0.95^{2x} = 18$

Recall that the balance of a fixed-rate interest earning account is given by the formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this equation, *A* is account balance, *P* is principal, *r* is annual interest rate, *t* is time in years, and *n* is the number of times interest is compounded each year.

29. Suppose you invest $12000 in a fixed-rate interest earning account, at 9% interest, compounded $n = 4$ times each year. Substitute the values given into the equation for compound interest above.

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30. When will the account contain \$20000? Solve your equation for the variable, *t*.

31. Suppose the account earned 12% interest, compounded *twice* a year. When would it contain \$20000?

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TAKE IT HOME

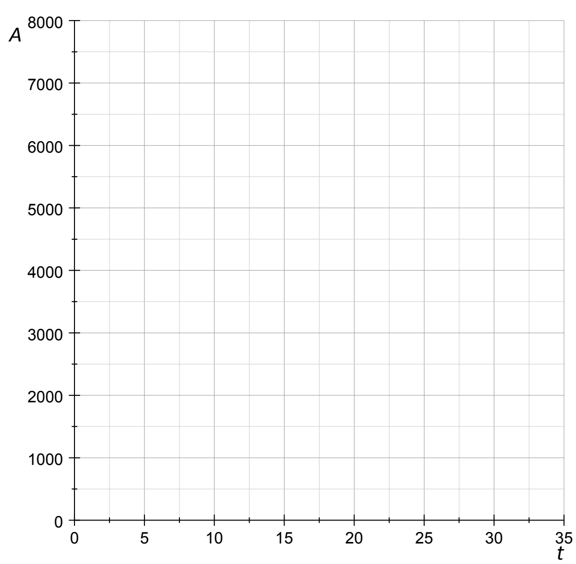
1 Art puts \$15500 in a bank account earning 6% interest. If the interest is compounded monthly (12 times a year) how much money will Art have in his account after 3 years?

2 Ronda has \$2300 in an investment earning 4% interest. If the interest is compounded semi-annually (two times a year), answer the following:

a. For each value of *t* given in the table, find the amount of money that Ronda will have in her account after that many years. Record your answers in the space provided in the table below.

Value of t	A
5	
10	
15	
20	
25	
30	

b. Use the values above to graph a curve that models the amount of money Ronda has in her account after *t* years.



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c. Use the graph in part (b) to approximate how many years it will take Ronda to have \$3500 in her account.

d. Use logarithms to find the number of years it will take Ronda to have \$3500 in her account. (See problems 29-31 in the lesson above)

3 Scott would like to buy a car in 4 years. He needs \$25000 to buy a new car. Currently Scott has \$19000 and he has decided to place the money in a bank account and earn interest in hopes that he will be able to buy his new car in 4 years. If he places the money in an account that earns 4% interest that compounds annually, will he reach his goal of \$25000 in four years?

4 Evaluate the following logarithms without the use of a calculator. (See #16-21 above)

a. $\log_3(3) =$

b. $\log_2(128) =$

c. $\log_4(64) =$

d. $\log_7(7^{12}) =$

e. $\log_5(5^9) =$

5 Evaluate the following logarithms using a calculator. (See #22-26 above)

a. $\log_2(91) =$

b. $\log_3(19) =$

c. $\log_7(213) =$

d. $\log_4(4^{34}) =$

e. $\log_2(2^{14}) =$

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6 Solve the following equations. (See Questions 27 & 28 in the above classroom activity.)

a. $6 \cdot 3.4^{2t} = 87$

b. $5 \cdot 9.1^{3x} = 130$

c. $4 \cdot 0.32^{2t} = 3$

d. $7 \cdot 0.97^{4x} = 4$

For the following problems, refer to Questions 29 through 31 from the classroom activity in this handout.

7 If you placed \$34000 in an account that earned 5% interest, compounded yearly, how much time would it take to have a balance of \$45000?

8 Akira has \$12000 to put in the bank. He found an account that earns 6% interest and is compounded quarterly. How long will Akira need to keep his money in that account if he wants to have a total of \$30000?

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9 Paula needs \$10000 for a down payment on a car; currently she has \$5000. Paula is going to deposit her money in an account that earns 7% interest and compounds monthly. How long will Paula need to keep her money in the account?

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