Inferential Statistics and Probability
a Holistic Approach

Chapter 6
Continuous Random Variables

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Continuous Distributions

- "Uncountable" Number of possibilities
- Probability of a point makes no sense
- Probability is measured over intervals
- Comparable to Relative Frequency Histogram – Find Area under curve.

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Discrete vs Continuous

- Countable
- Discrete Points
- p(x) is probability distribution function
- $p(x) \geq 0$
- $\Sigma p(x) = 1$

- Uncountable
- Continuous Intervals
- f(x) is probability density function
- $f(x) \geq 0$
- Total Area under curve =1

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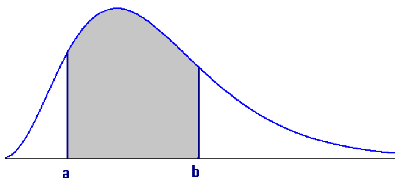
Continuous Random Variable

- f(x) is a density function
- P(X<x) is a distribution function.
- P(a<X<b) = area under function between a and b



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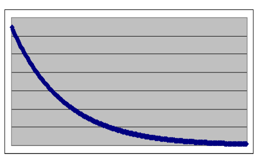
Exponential distribution

- Waiting time
- "Memoryless"
- $f(x) = (1/\mu)e^{-(1/\mu)x}$
- $P(x>a) = e^{-(a/\mu)}$
- $\mu=\mu \quad \sigma^2=\mu^2$
- $P(x>a+b|x>b) = e^{-(a/\mu)}$



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Examples of Exponential Distributiuon

- Time until…
- a circuit will fail
- the next RM 7 Earthquake
- the next customer calls
- An oil refinery accident
- you buy a winning lotto ticket

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Relationship between Poisson and Exponential Distributions

- If occurrences follow a Poisson Process with mean = μ, then the waiting time for the next occurrence has Exponential distribution with mean = $1/\mu$.
- Example: If accidents occur at a plant at a constant rate of 3 per month, then the expected waiting time for the next accident is 1/3 month.

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Exponential Example

The time until a screen is cracked on a smart phone has exponential distribution with μ=500 hours of use.

(a) Find the probability screen will not crack for at least 600 hours.

$P(x>600) = e^{-600/500} = e^{-1.2} = .3012$

(b) Assuming that screen has already lasted 500 hours without cracking, find the chance the display will last an additional 600 hours.

$P(x>1100|x>500) = P(x>600) = .3012$

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Exponential Example

The time until a screen is cracked on a smart phone has exponential distribution with μ=500 hours of use.

(a) Find the median of the distribution

$P(x>med) = e^{-(med)/500} = 0.5$

$med = -500\ln(.5) = 347$

p^{th} Percentile = $-\mu \ln(1-p)$

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Uniform Distribution

- Rectangular distribution
- Example: Random number generator

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$

$$\mu = E(X) = \frac{b+a}{2}$$

$$\sigma^2 = Var(X) = \frac{(b-a)^2}{12}$$

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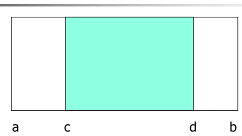
Uniform Distribution - Probability



$$P(c < X < d) = \frac{d-c}{b-a}$$

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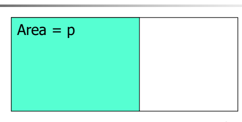
Uniform Distribution - Percentile



Formula to find the pth percentile X_p:

$$X_p = a + p(b-a)$$

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Uniform Example 1

- Find mean, variance, P(X<3) and 70th percentile for a uniform distribution from 1 to 11.

$$\mu = \frac{1+11}{2} = 6 \quad \sigma^2 = \frac{(11-1)^2}{12} = 8.33$$

$$P(X<3) = \frac{3-1}{11-1} = 0.3$$

$$X_{70} = 1 + 0.7(11-1) = 8$$

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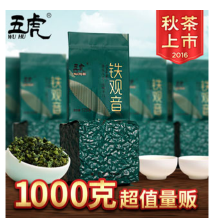
Uniform Example 2

- A tea lover orders 1000 grams of Tie Guan Yin loose leaf when his supply gets to 50 grams.
- The amount of tea currently in stock follows a uniform random variable.
- Determine this model
- Find the mean and variance
- Find the probability of at least 700 grams in stock.
- Find the 80th percentile



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Uniform Example 3

- A bus arrives at a stop every 20 minutes.
  - Find the probability of waiting more than 15 minutes for the bus after arriving randomly at the bus stop.
  - If you have already waited 5 minutes, find the probability of waiting an additional 10 minutes or more. (Hint: recalculate parameters a and b)

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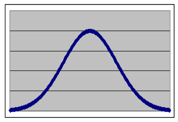
Normal Distribution

- The normal curve is *bell-shaped*
- The mean, median, and mode of the distribution are equal and located at the peak.
- The normal distribution is symmetrical about its mean. Half the area under the curve is above the peak, and the other half is below it.
- The normal probability distribution is asymptotic - the curve gets closer and closer to the x-axis but never actually touches it.



$$f(x) = \frac{e^{-\frac{1}{2\sigma}(x-\mu)^2}}{\sigma\sqrt{2\pi}}$$

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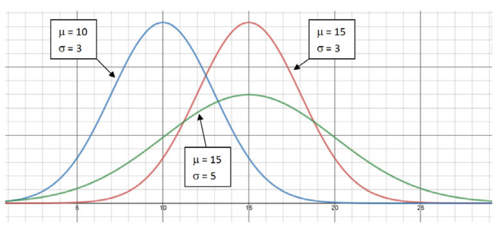
Examples of Normal Random Variables



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The Standard Normal Probability Distribution

- A normal distribution with a mean of 0 and a standard deviation of 1 is called the standard normal distribution.
- Z value: The distance between a selected value, designated x, and the population mean μ, divided by the population standard deviation, σ

$$Z = \frac{X - \mu}{\sigma}$$

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Areas Under the Normal Curve – Empirical Rule

- About 68 percent of the area under the normal curve is within one standard deviation of the mean. $\mu \pm 1\sigma$
- About 95 percent is within two standard deviations of the mean $\mu \pm 2\sigma$
- 99.7 percent is within three standard deviations of the mean. $\mu \pm 3\sigma$

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EXAMPLE

- The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.
- About 68% of the daily water usage per person in New Providence lies between what two values?
- $\mu \pm 1\sigma = 20 \pm 1(5)$. That is, about 68% of the daily water usage will lie between 15 and 25 gallons.

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Normal Distribution – probability problem procedure

- Given: Interval in terms of X
- Convert to Z by $Z = \frac{X - \mu}{\sigma}$
- Look up probability in table or technology

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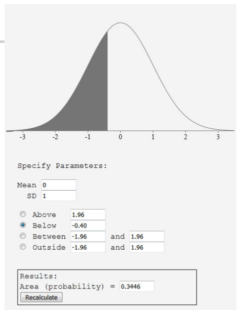
EXAMPLE

- The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.
- What is the probability that a person from the town selected at random will use less than 18 gallons per day?
- The associated Z value is Z=(18-20)/5=0.
- P(X<18)=P(Z<-0.40)=**0.3446**



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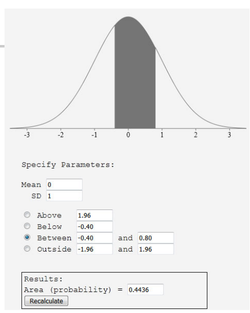
EXAMPLE

- The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.
- What proportion of the people uses between 18 and 24 gallons?
- The Z value associated with x=18, Z = (18-20)/5 = -0.40 x=24, Z = (24-20)/5 = 0.80.
- P(18<X<24)= P(-0.40<Z<0.80) = .7881. – 3446 = **0.4435**



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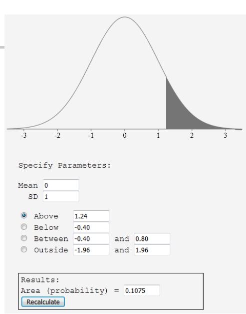
EXAMPLE

- The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.
- What percentage of the population uses more than 26.2 gallons?
- The Z value associated with X=26.2, Z=(26.2-20)/5=1.24.
- P(X>26.2)=P(Z>1.24) =1-.8925= **0.1075**



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Normal Distribution – percentile problem procedure

- Given: probability or percentile desired.
- Use table or technology that corresponds to probability to get Z
- Convert to X by the formula:

$$X = \mu + Z\sigma$$

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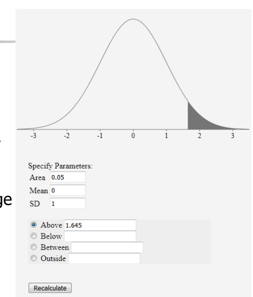
EXAMPLE

- The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons. A special tax is going to be charged on the top 5% of water users.
- Find the value of daily water usage that generates the special tax
- The Z value associated with 95th percentile =1.645
- X=20 + 5(1.645)
  = **28.2 gallons per day**



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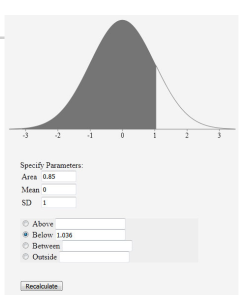
EXAMPLE

- Professor Kurv has determined that the final averages in his statistics course is normally distributed with a mean of 77.1 and a standard deviation of 11.2.
- He decides to assign his grades for his current course such that the top 15% of the students receive an A.
- What is the lowest average a student can receive to earn an A?
- The top 15% would be the finding the 85th percentile.
- The corresponding Z value is 1.036
- Thus we have X=77.1+(1.036)(11.2), or **X=88.7**



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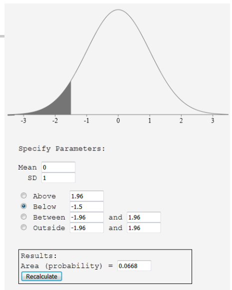
EXAMPLE

The amount of tip the servers in an exclusive restaurant receive per shift is normally distributed with a mean of \$80 and a standard deviation of \$10.

- Shelli feels she has had a bad shift if her total tip for the shift is less than \$65.
- What percentage of the time will she feel like she provided poor service?
- Let y be the amount of tip. $X=65$, $Z= (65-80)/10= -1.5$.
- $P(X<65)=P(Z<-1.5)=$**0.0668**



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