

Stat 6863-Handout 6

Life Insurance & Annuities - General

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In the prior handouts, we discussed amount of benefits, survival time until an event and interest as random variables, although we assumed for simplicity that interest was deterministic. These components make up the general models of insurance and annuities used in actuarial science.

Insurances Payable at the Moment of Death

In this model, the amount (b_T) and time of payment (T) will be dependent on time of death. The discount random variable of interest (v_T) is also dependent on t . The present value (Z) at policy issue of the benefit payment can be shown as:

$$Z = b_T v_T$$

n-year Term Life Insurance

In this model, a benefit is provided only if death occurs before the passage of n years from policy issuance. For each unit of benefit:

$$b_t = \begin{cases} 1 & t \leq n \\ 0 & t > n \end{cases} \quad v_t = v^t \quad Z = \begin{cases} v^T & t \leq n \\ 0 & t > n \end{cases}$$

The expected value of the present value random variable Z is called the **net single premium**. *Net* means the premium has not been loaded for expenses and *single* means the premium is paid all at once, rather than in periodic payments, which is more typical. In pension plans, this net single premium is usually referred to as the **actuarial present value**. For n -year term life insurance of benefit 1 issued at age x , the actuarial present value payable at the instant of death is denoted by:

$$\bar{A}_{x:n|}^1 = E(Z) = \int_0^n v^t f_x(t) dt = \int_0^n e^{-\delta t} {}_t p_x \mu_{x+t} dt$$

Whole Life Insurance

Whole life insurance is the limiting case of n -year term life insurance as $n \rightarrow \infty$. provides a benefit if death occurs any time in the future. The random variable and net single premium assuming payment at the instant of death are shown:

$$b_t = 1 \quad v_t = v^t = e^{-\delta t} \quad Z = v^T$$

$$\bar{A}_x = E(Z) = \int_0^\infty v^t {}_t p_x \mu_{x+t} dt$$

Example 1:

Assume the PDF of the future lifetime is uniform where:

$$f(t) = \frac{1}{100} \quad 0 \leq t \leq 100$$

Find the net single premium of a whole life policy issued at age 0 when the force of interest (δ) = .08 and face value of the policy is \$1000:

$$1000\bar{A}_0 = 1000 \int_0^{100} e^{-.08t} \frac{1}{100} dt = 1000 \left(\frac{1 - e^{-8}}{8} \right) = 124.96 \quad (\text{note: } v^t = e^{-\delta t})$$

n-year Pure Endowment

The n-year pure endowment pays a deferred benefit if the holder of this policy is still alive after n years. For the individual age x, the present value of 1 unit payable is written as:

$${}_nE_x = v^n {}_np_x$$

Note we have changed our notation as endowment payments are used more in pension plans rather than insurance plans.

Example 2:

An executive compensation plan pays a lump sum benefit of \$100,000 to participants who do not terminate employment in the next 5 years. Find the present value of this benefit for a 50-year old executive using 7.5% interest. Assume that ${}_5p_{50}$ for all causes is 75%.

$$100000 {}_5E_{50} = 100000 \left(\frac{1}{1.075} \right)^5 .75 = 52242$$

Life Annuities

From theory of interest, we calculated the annuity payable at the beginning of each period (annuity-due) and the annuity payable at the end of each period (annuity-immediate). The present value for these annuities payable for n years were shown as:

$$\ddot{a}_{n|} = \sum_{i=1}^n v^{i-1} \quad a_{n|} = \sum_{i=1}^n v^i$$

Now instead of a term-certain annuity-due, take an annuity where one unit of payment was made to a participant age x at the beginning of each year, assuming the participant was alive. This is the same as a series of endowment payments for 0,1, 2, 3, ... years so the present value can be written as

$$1 + {}_1E_x + {}_2E_x + \dots \quad \text{where} \quad {}_kE_x = v^k {}_kp_x$$

This stream of endowment payments is called a **straight life annuity** and is the root benefit for pension funds. The present value forms for the straight life annuity-due and the straight life annuity-immediate are shown:

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x \quad a_x = \sum_{k=1}^{\infty} v^k {}_kp_x$$

It should be clear that $\ddot{a}_x - a_x = 1$.

Sometimes, life annuities may be limited to a fixed period of time. This is an **n-year temporary life annuity** and can be thought of as a fixed number of consecutive endowment contracts. The present value forms are shown:

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x \quad a_{x:\overline{n}|} = \sum_{k=1}^n v^k {}_kp_x$$

Finally, a **deferred life annuity** for an individual age x is a straight life annuity commencing at age $x+n$, assuming x is still alive at initial benefit commencement. In reality, this is simply an n -year endowment of an annuity! The present value of the deferred annuity-due and deferred annuity-immediate are shown:

$${}_n|\ddot{a}_x = {}_nE_x \ddot{a}_{x+n} = \sum_{k=n}^{\infty} v^k {}_kp_x \quad {}_n|a_x = {}_nE_x a_{x+n} = \sum_{k=n+1}^{\infty} v^k {}_kp_x$$

These annuity functions can all be related by the following intuitive equation:

$$\ddot{a}_x = {}_n|\ddot{a}_x + \ddot{a}_{x:\overline{n}|}$$

These topics will be explored more in the next section where we combine the interest discounts with the values of the life table.

Life Annuities with *mt*hly Payments

In practice, annuities are paid in periods of time shorter than one year, usually monthly or quarterly. Analogous to the term certain model, we denote the actuarial present value of a life annuity-due of 1 paid *m*thly in units of $1/m$ to an individual age x as $\ddot{a}_x^{(12)}$. If we use the uniform distribution of death between integer ages, this straight life annuity can be approximated as:

$$\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{m-1}{2m}$$

Similar notation is available for the other annuity forms. However, with the use of spreadsheet programs it is possible to model almost any annuity imaginable and calculate the present value directly.

Commutation Functions

In working with the life table, it is often easier to use commutation functions to determine annuities. Although the text says these types of deterministic functions are going out of favor, they are still used frequently in the pension area. The formulas for the most popular commutation functions are shown below:

Function	Use
$D_x = v^x l_x$	Insurance and Annuities
$C_x = v^{x+1} d_x$	Insurance
$M_x = \sum_{k=0}^{\infty} C_{x+k}$	Insurance
$N_x = \sum_{k=0}^{\infty} D_{x+k}$	Annuities

We will focus on the annuity examples which are used in the pension plan environment, where we will use D_x and N_x to build annuities.

The endowment function ${}_tE_x$ can be shown as the ratio of D_x functions:

$${}_tE_x = v^t {}_tP_x = \frac{v^{x+t}}{v^x} \cdot \frac{l_{x+t}}{l_x} = \frac{D_{x+t}}{D_x}$$

The straight life annuity is simply a ratio of the N commutation function to the D function:

$$\ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{\infty} \frac{D_{x+k}}{D_x} = \frac{N_x}{D_x}$$

The deferred life annuity and the temporary annuity can also be written as commutation functions:

$${}_n|\ddot{a}_x = {}_nE_x \ddot{a}_{x+n} = \frac{D_{x+n}}{D_x} \cdot \frac{N_{x+n}}{D_{x+n}} = \frac{N_{x+n}}{D_x}$$

$$\ddot{a}_{x:n|} = \ddot{a}_x - {}_n|\ddot{a}_x = \frac{N_x - N_{x+n}}{D_x}$$

Adjustment for m thly annuities:

D_x can be adjusted for non-integer values of x by methods of interpolation. If we assume a uniform distribution of deaths, this interpolation is linear and:

$$N_x^{(m)} \cong N_x - \frac{m-1}{2m} D_x$$

Example: A life table is shown below

x	q_x	l_x	d_x	$v_x(8\%)$	D_x	N_x	$N_x^{(12)}$	$\ddot{a}_x^{(x)}$
0	0.25	100000	25000	1	100000	269661.35	223828.02	2.238
1	0.2	75000	15000	0.92593	69444.444	169661.35	137832.65	1.985
2	0.3	60000	18000	0.85734	51440.329	100216.91	76640.093	1.490
3	0.5	42000	21000	0.79383	33340.954	48776.581	33495.31	1.005
4	1	21000	21000	0.73503	15435.627	15435.627	8360.9646	0.542

Find the present value of (1) a monthly annuity of 500 commencing at age 1 and payable for two years and (2) a deferred life annuity of 750 commencing at age 1.75 for an individual age 0:

$$(6000)\ddot{a}_{x:\overline{2}|}^{(12)} = 6000 \frac{137832.65 - 33495.31}{69444.444} = 9014.75$$

$$N_{1.75} = (.25)(169661.35) + (.75)(100216.91) = 117578.02$$

$$(9000)_{1.75}\ddot{a}_0^{(12)} = 9000 \frac{117578.02}{100000} = 10582.02$$

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Relationship between Annuities and Insurance

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Insurances Payable at the End of the year of Death

Although in practice insurance is payable at the moment of death, the life table is based on annual census data. Therefore discrete formulas are needed for net single premiums of insurances payable at the end of the year of death:

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_k p_x q_{x+k} = \sum_{k=0}^{\infty} \frac{v^{k+1} d_{x+k}}{l_x} \quad A_{x:\overline{n}|}^1 = \sum_{k=0}^{n-1} v^{k+1} {}_k p_x q_{x+k} = \sum_{k=0}^{n-1} \frac{v^{k+1} d_{x+k}}{l_x}$$

Discrete Insurance as a Function of Discrete Annuities

The difference between the present value of a life annuity payable at the beginning of the year ($\ddot{a}_x$) and the present value of a life annuity payable at the end of the year (a_x) is (1) the final payment and (2) one year of interest on each year's payment. If the payments of $\ddot{a}_x$ are adjusted so the interest is equivalent for each payment of a_x , then the present value of the missing payment for a_x has the same present value as the net single premium for an insurance payable at the end of the year of death, $A_x = v\ddot{a}_x - a_x$.

$$\begin{aligned} v\ddot{a}_x - a_x &= v[1 + v {}^1_1 p_x + v^2 {}^2_2 p_x + \dots] - [v {}^1_1 p_x + v^2 {}^2_2 p_x + v^3 {}^3_3 p_x + \dots] \\ &= \left[v {}^1 \frac{l_x}{l_x} + v^2 {}^2 \frac{l_{x+1}}{l_x} + v^3 {}^3 \frac{l_{x+2}}{l_x} + \dots \right] - \left[v {}^1 \frac{l_{x+1}}{l_x} + v^2 {}^2 \frac{l_{x+2}}{l_x} + v^3 {}^3 \frac{l_{x+3}}{l_x} + \dots \right] \\ &= v {}^1 \frac{l_x - l_{x+1}}{l_x} + v^2 {}^2 \frac{l_{x+1} - l_{x+2}}{l_x} + v^3 {}^3 \frac{l_{x+2} - l_{x+3}}{l_x} + \dots \\ &= v {}^1 \frac{d_x}{l_x} + v^2 {}^2 \frac{d_{x+1}}{l_x} + v^3 {}^3 \frac{d_{x+2}}{l_x} + \dots \\ &= A_x \end{aligned}$$

So $A_x = v\ddot{a}_x - a_x$ and $A_{x:\overline{n}|}^1 = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}$

Also, $A_x = v\ddot{a}_x - a_x = v\ddot{a}_x - (\ddot{a}_x - 1) = 1 + (v-1)\ddot{a}_x = 1 + \left(\frac{1}{1+i} - 1\right)\ddot{a}_x = 1 - \left(\frac{i}{1+i}\right)\ddot{a}_x = 1 - d\ddot{a}_x$

So $A_x = 1 - d\ddot{a}_x$

Example: If $\ddot{a}_{50} = 13$ and $i=6.5\%$, find A_{50} :

$$A_{50} = 1 - \frac{.065}{1.065} 13 = .206573$$

Adjustment for Insurances Payable at the Moment of Death

$\delta = \ln(1+i)$ is the force of interest used if annuities are paid continuously. If the force of mortality is constant over 1 year (Uniform Distribution of Deaths), then the life annuities and insurance can be easily adjusted from end of year formulas to continuous formulas:

$$\bar{a}_x \approx \frac{i}{\delta} a_x$$

$$\bar{a}_{x:\overline{n}|} \approx \frac{i}{\delta} a_{x:\overline{n}|}$$

$${}_n|\bar{a}_x \approx \frac{i}{\delta} {}_n|a_x$$

Since the insurances are really functions of annuities

$$A_x = 1 - d\ddot{a}_x$$

the same approximations apply (exact under UDD):

$$\bar{A}_x = 1 - \delta\bar{a}_x$$

$$\bar{A}_x = \frac{i}{\delta} A_x \quad \text{and} \quad \bar{A}_{x:\overline{n}|}^1 = \frac{i}{\delta} A_{x:\overline{n}|}^1$$

Example: You are given $i = .065$ $\ddot{a}_{50:\overline{10}|} = 7$ $a_{50:\overline{10}|} = 6.5$. Find the net single premium for a \$250,000 10 year term insurance policy issued to age 50 payable at the moment of death.

$$\begin{aligned} NSP &= \$250,000 \bar{A}_{50:\overline{10}|}^1 \\ &= \$250,000 \frac{i}{\delta} (\ddot{a}_{50:\overline{10}|} - a_{50:\overline{10}|}) \\ &= \$250,000 \frac{.065}{\ln(1.065)} \left[\frac{7}{1.065} - 6.5 \right] \\ &= \$18,778 \end{aligned}$$