Continuous Distributions

- “Uncountable” Number of possibilities
- Probability of a point makes no sense
- Probability is measured over intervals
- Comparable to Relative Frequency
  Histogram – Find Area under curve.

Discrete vs Continuous

- Countable
  - Discrete Points
  - \( p(x) \) is probability distribution function
    - \( p(x) \geq 0 \)
    - \( \Sigma p(x) = 1 \)
- Uncountable
  - Continuous Intervals
  - \( f(x) \) is probability density function
    - \( f(x) \geq 0 \)
    - Total Area under curve = 1
Continuous Random Variable

- $f(x)$ is a density function
- $P(X<x)$ is a distribution function.
- $P(a<X<b) = \text{area under function between } a \text{ and } b$

Exponential distribution

- Waiting time
- “Memoryless”
- $f(x) = \frac{1}{\mu}e^{-\frac{1}{\mu}x}$
- $P(x>a) = e^{-\frac{a}{\mu}}$
- $\mu = \mu$, $\sigma^2 = \mu^2$
- $P(x>a+b|x>b) = e^{-\frac{a}{\mu}}$

Examples of Exponential Distribution

- Time until...
- a circuit will fail
- the next RM 7 Earthquake
- the next customer calls
- An oil refinery accident
- you buy a winning lotto ticket
Relationship between Poisson and Exponential Distributions

- If occurrences follow a Poisson Process with mean $= \mu$, then the waiting time for the next occurrence has Exponential distribution with mean $= 1/\mu$.

- Example: If accidents occur at a plant at a constant rate of 3 per month, then the expected waiting time for the next accident is $1/3$ month.

Exponential Example

The life of a digital display of a calculator has exponential distribution with $\mu=500$ hours.

- (a) Find the chance the display will last at least 600 hours.

\[ P(x>600) = e^{-600/500} = e^{-1.2} = .3012 \]

- (b) Assuming it has already lasted 500 hours, find the chance the display will last an additional 600 hours.

\[ P(x>1100|x>500) = P(x>600) = .3012 \]

Exponential Example

The life of a digital display of a calculator has exponential distribution with $\mu=500$ hours.

- (a) Find the median of the distribution

\[ P(x>\text{med}) = e^{-\text{med}/500} = 0.5 \]

\[ \text{med} = -\ln(0.5) \times 500 = 346.57 \]

- $p^{th}$ Percentile $= -\ln(1-p)\mu$
**Uniform Distribution**

- Rectangular distribution
- Example: Random number generator

\[ f(x) = \frac{1}{b-a} \quad a \leq x \leq b \]

\[ \mu = E(X) = \frac{b+a}{2} \]

\[ \sigma^2 = Var(X) = \frac{(b-a)^2}{12} \]

**Uniform Distribution - Probability**

\[ P(c < X < d) = \frac{d-c}{b-a} \]

**Uniform Distribution - Percentile**

Formula to find the pth percentile \(X_p\):

\[ X_p = a + p(b-a) \]
Uniform Example 1

- Find mean, variance, $P(X<3)$ and 70th percentile for a uniform distribution from 1 to 11.
  \[
  \mu = \frac{1 + 11}{2} = 6, \quad \sigma^2 = \frac{(11-1)^2}{12} = 8.33
  \]
  \[
  P(X < 3) = \frac{3 - 1}{11 - 1} = 0.3
  \]
  \[
  X_{70} = 1 + 0.7(11-1) = 8
  \]

Uniform Example 2

- A tea lover orders 1000 grams of Tie Guan Yin loose leaf when his supply gets to 50 grams.
- The amount of tea currently in stock follows a uniform random variable.
- Determine this model
- Find the mean and variance
- Find the probability of at least 700 grams in stock.
- Find the 80th percentile

Uniform Example 3

- A bus arrives at a stop every 20 minutes.
  - Find the probability of waiting more than 15 minutes for the bus after arriving randomly at the bus stop.
  - If you have already waited 5 minutes, find the probability of waiting an additional 10 minutes or more. (Hint: recalculate parameters $a$ and $b$)
Normal Distribution

- The normal curve is bell-shaped.
- The mean, median, and mode of the distribution are equal and located at the peak.
- The normal distribution is symmetrical about its mean. Half the area under the curve is above the peak, and the other half is below it.
- The normal probability distribution is asymptotic - the curve gets closer and closer to the x-axis but never actually touches it.

The normal distribution is given by the equation:

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

The Standard Normal Probability Distribution

- A normal distribution with a mean of 0 and a standard deviation of 1 is called the standard normal distribution.
- Z value: The distance between a selected value, designated x, and the population mean \( \mu \), divided by the population standard deviation, \( \sigma \)

\[ Z = \frac{X - \mu}{\sigma} \]

Areas Under the Normal Curve - Empirical Rule

- About 68 percent of the area under the normal curve is within one standard deviation of the mean, \( \mu \pm 1\sigma \)
- About 95 percent is within two standard deviations of the mean, \( \mu \pm 2\sigma \)
- 99.7 percent is within three standard deviations of the mean, \( \mu \pm 3\sigma \)
EXAMPLE

The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.

About 68% of the daily water usage per person in New Providence lies between what two values?

\[ \mu \pm 1\sigma = 20 \pm 5 \]

That is, about 68% of the daily water usage will lie between 15 and 25 gallons.

Normal Distribution – probability problem procedure

- Given: Interval in terms of \( X \)
- Convert to \( Z \) by \[ Z = \frac{X - \mu}{\sigma} \]
- Look up probability in table.

EXAMPLE

The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.

What is the probability that a person from the town selected at random will use less than 18 gallons per day?

The associated \( Z \) value is \[ Z = \frac{18 - 20}{5} = -0.40 \]

Thus, \( P(X < 18) = P(Z < -0.40) = 0.3446 \)
EXAMPLE continued

The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.

What proportion of the people uses between 18 and 24 gallons?

The Z value associated with x=18 is Z=-0.40 and with x=24, Z=(24-20)/5=0.80.

Thus, \( P(18 < X < 24) = P(-0.40 < Z < 0.80) = 0.7881 - 0.3446 = 0.4435 \)

EXAMPLE continued

The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons.

What percentage of the population uses more than 26.2 gallons?


Thus \( P(X > 26.2) = P(Z > 1.24) = 1 - 0.8925 = 0.1075 \)

Normal Distribution – percentile problem procedure

- Given: probability or percentile desired.
- Look up Z value in table that corresponds to probability.
- Convert to X by the formula:

\[ X = \mu + Z\sigma \]
EXAMPLE

The daily water usage per person in a town is normally distributed with a mean of 20 gallons and a standard deviation of 5 gallons. A special tax is going to be charged on the top 5% of water users.

- Find the value of daily water usage that generates the special tax
- The Z value associated with 95th percentile = 1.645
- $X = 20 + 5(1.645) = 28.2$ gallons per day

EXAMPLE

Professor Kurv has determined that the final averages in his statistics course is normally distributed with a mean of 77.1 and a standard deviation of 11.2.

- He decides to assign his grades for his current course such that the top 15% of the students receive an A.
- What is the lowest average a student can receive to earn an A?
- The top 15% would be the finding the 85th percentile. Find k such that $P(X < k) = .85$.
- The corresponding Z value is 1.04. Thus we have $X = 77.1 + (1.04)(11.2)$, or $X = 88.75$

EXAMPLE

The amount of tip the servers in an exclusive restaurant receive per shift is normally distributed with a mean of $80 and a standard deviation of $10.

- Shelli feels she has provided poor service if her total tip for the shift is less than $65.
- What percentage of the time will she feel like she provided poor service?

- Let $y$ be the amount of tip. The Z value associated with $X = 65$ is $Z = (65-80)/10 = -1.5$.
- Thus $P(X < 65) = P(Z < -1.5) = .0668$. 
Distribution of Sample Mean

- Random Sample: $X_1, X_2, X_3, ..., X_n$
  - Each $X_i$ is a Random Variable from the same population
  - All $X_i$'s are Mutually Independent
- $\bar{X}$ is a function of Random Variables, so $\bar{X}$ is itself Random Variable.
- In other words, the Sample Mean can change if the values of the Random Sample change.
- What is the Probability Distribution of $\bar{X}$?

Example – Roll 1 Die

```
<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.16</td>
</tr>
<tr>
<td>2</td>
<td>0.18</td>
</tr>
</tbody>
</table>
```

Example – Roll 2 Dice

```
<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.16</td>
</tr>
<tr>
<td>3</td>
<td>0.18</td>
</tr>
</tbody>
</table>
```
Example – Roll 10 Dice

Example – Roll 30 Dice

Example - Poisson

μ = 1.25
μ = 12.5
Central Limit Theorem – Part 1

IF a Random Sample of **any size** is taken from a population with a **Normal Distribution** with mean $= \mu$ and standard deviation $= \sigma$

THEN the distribution of the sample mean has a Normal Distribution with:

$$\mu_X = \mu \quad \sigma_X = \frac{\sigma}{\sqrt{n}}$$

Central Limit Theorem – Part 2

IF a random sample of **sufficiently large size** is taken from a population with **any Distribution** with mean $= \mu$ and standard deviation $= \sigma$

THEN the distribution of the sample mean has approximately a Normal Distribution with:

$$\mu_X = \mu \quad \sigma_X = \frac{\sigma}{\sqrt{n}}$$

Central Limit Theorem

3 important results for the distribution of $X$

- Mean Stays the same
  $$\mu_X = \mu$$

- Standard Deviation Gets Smaller
  $$\sigma_X = \frac{\sigma}{\sqrt{n}}$$

- If n is sufficiently large, $X$ has a Normal Distribution
Example

The mean height of American men (ages 20-29) is \( \mu = 69.2 \) inches. If a random sample of 60 men in this age group is selected, what is the probability the mean height for the sample is greater than 70 inches? Assume \( \sigma = 2.9 \) inches.

\[
P(\bar{X} > 70) = P \left( Z > \frac{70 - 69.2}{2.9/\sqrt{60}} \right) = P(Z > 2.14) = 0.0162
\]

Example (cont)

\[\mu = 69.2 \quad \sigma = 2.9\]

Example - Central Limit Theorem

The waiting time until receiving a text message follows an exponential distribution with an expected waiting time of 1.5 minutes. Find the probability that the mean waiting time for the 50 text messages exceeds 1.6 minutes.

\[
\mu = 1.5 \quad \sigma = 1.5 \quad n = 40
\]

Use Normal Distribution \((n>30)\)

\[
P(\bar{X} > 1.6) = P \left( Z > \frac{1.6 - 1.5}{1.5/\sqrt{50}} \right) = P(Z > 0.47) = 0.3192
\]