Comparing two population means

Four models
- Independent Sampling
  - Large Sample or known variances
    - Z-test
  - The 2 population variances are equal
    - Pooled variance t-test
  - The 2 population variances are unequal
    - t-test for unequal variances
- Dependent Sampling
  - Matched Pairs t-test

Independent Sampling

Population 1
- \( \mu_1, \sigma_1 \)
- \( \bar{x}_1, s_1 \)

Population 2
- \( \mu_2, \sigma_2 \)
- \( \bar{x}_2, s_2 \)
Dependent sampling

Difference of Two Population means

- $\bar{x}_1 - \bar{x}_2$ is Random Variable
- $\bar{x}_1 - \bar{x}_2$ is a point estimator for $\mu_1 - \mu_2$
- The standard deviation is given by the formula $\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
- If $n_1$ and $n_2$ are sufficiently large, $\bar{x}_1 - \bar{x}_2$ follows a normal distribution.

Difference between two means - large sample Z test

- If both $n_1$ and $n_2$ are over 30 and the two populations are independently selected, this test can be run.
- Test Statistic:
  \[ Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \]
Example 1

- Are larger houses more likely to have pools?
- The housing data square footage (size) was split into two groups by pool (Y/N).
- Test the hypothesis that the homes with pools have more square feet than the homes without pools. Let $\alpha = .01$

Example 1 - Design

$H_0: \mu_1 \leq \mu_2 \quad H_a: \mu_1 > \mu_2$

$H_0: \mu_1 - \mu_2 \leq 0 \quad H_a: \mu_1 - \mu_2 > 0$

$\alpha = .01$

$Z = (\bar{X}_1 - \bar{X}_2) / (\sqrt{\sigma_1^2 / n_1 + \sigma_2^2 / n_2})$

$H_0$ is rejected if $Z > 2.326$

Example 1 - Data

- Population 1
  - Size with pool
  - Sample size = 130
  - Sample mean = 26.25
  - Standard Dev = 6.93
- Population 2
  - Size without pool
  - Sample size = 95
  - Sample mean = 23.04
  - Standard Dev = 4.55
EXAMPLE 1 - DATA

\[ Z = \frac{(26.25 - 23.04) - 0}{\sqrt{\frac{6.93^2}{130} + \frac{4.55^2}{95}}} = 4.19 \]

- Decision: Reject Ho
- Conclusion: Homes with pools have more mean square footage.

EXAMPLE 1 - p-value method

- Using Technology
  - Reject Ho if the p-value < \( \alpha \)

<table>
<thead>
<tr>
<th></th>
<th>Sq ft with pool</th>
<th>Sq ft no pool</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>26.25</td>
<td>23.04</td>
</tr>
<tr>
<td>Std Dev</td>
<td>6.93</td>
<td>4.55</td>
</tr>
<tr>
<td>Observations</td>
<td>130</td>
<td>95</td>
</tr>
<tr>
<td>Hypothesized Mean Difference</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Z</td>
<td>4.19</td>
<td></td>
</tr>
<tr>
<td>p-value</td>
<td>0.0000137</td>
<td></td>
</tr>
</tbody>
</table>

EXAMPLE 1 - Results/Decision

- Test Statistic = 4.19
- p-value = 0.0000137
- Critical Value = 2.326
- Decision: Reject Ho
To conduct this test, three assumptions are required:
- The populations must be normally or approximately normally distributed (or central limit theorem must apply).
- The sampling of populations must be independent.
- The population variances must be equal.

**Pooled Sample Variance and Test Statistic**

- Pooled Sample Variance: 
  \[ s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} \]
- Test Statistic: 
  \[ t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \]
  \[ df = n_1 + n_2 - 2 \]

**EXAMPLE 2**

A recent EPA study compared the highway fuel economy of domestic and imported passenger cars.
- A sample of 12 imported cars revealed a mean of 35.76 mpg with a standard deviation of 3.86.
- A sample of 15 domestic cars revealed a mean of 33.59 mpg with a standard deviation of 2.16 mpg.
- At the .05 significance level can the EPA conclude that the mpg is higher on the imported cars? (Let subscript 2 be associated with domestic cars.)
EXAMPLE 2

- $H_0 : \mu_1 \leq \mu_2$  
- $H_a : \mu_1 > \mu_2$
- $\alpha = 0.05$
- $t = (\bar{X}_1 - \bar{X}_2) / (s_p \sqrt{1/n_1 + 1/n_2})$
- $H_0$ is rejected if $t > 1.708$, $df=25$
- $t=1.85$ $H_0$ is rejected. Imports have a higher mean mpg than domestic cars.

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EXAMPLE 2

- $H_0 : \mu_1 \leq \mu_2$  
- $H_a : \mu_1 > \mu_2$
- $\alpha = 0.05$
- $t$ test
- $H_0$ is rejected if $t > 1.746$, $df=16$
- $t'=1.74$ $H_0$ is not rejected. There is insufficient sample evidence to claim a higher mpg on the imported cars.
Using Technology

- Decision Rule: Reject $H_0$ if $p$-value $< \alpha$
- Megastat: Compare Two Independent Groups
- Use Equal Variance or Unequal Variance Test
- Use Original Data or Summarized Data

```plaintext
domestic: 29.8 33.3 34.7 37.4 34.4 32.7 30.2 36.2 35.5 34.6 33.2 35.1 33.6 31.3 31.9
import: 39.0 35.1 39.1 32.2 35.6 35.5 40.8 34.7 33.2 29.4 42.3 32.2
```

Megastat Result – Equal Variances

```plaintext
domestic:
35.76
33.59

import:
39.0
35.1
39.1
32.2
35.6
35.5
40.8
34.7
33.2
29.4
42.3
32.2

2.17000 difference (import - domestic)
9.18858 pooled variance
3.02956 pooled std. dev.
1.17273 standard error of difference
0 hypothesized difference
1.85 t
0.0381 p-value (one-tailed, upper)
```

Megastat Result – Unequal Variances

```plaintext
domestic:
35.76
33.59

import:
39.0
35.1
39.1
32.2
35.6
35.5
40.8
34.7
33.2
29.4
42.3
32.2

16 df
2.17000 difference (import - domestic)
1.24606 standard error of difference
0 hypothesized difference
1.74 t
0.0594 p-value (one-tailed, upper)
```
Hypothesis Testing - Paired Observations

- Independent samples are samples that are not related in any way.
- Dependent samples are samples that are paired or related in some fashion.
  - For example, if you wished to buy a car you would look at the same car at two (or more) different dealerships and compare the prices.
- Use the following test when the samples are dependent:

Hypothesis Testing Involving Paired Observations

\[ t = \frac{\bar{X}_d - \mu_d}{s_d / \sqrt{n}} \]

- where \( \bar{X}_d \) is the average of the differences
- \( s_d \) is the standard deviation of the differences
- \( n \) is the number of pairs (differences)

EXAMPLE 3

- An independent testing agency is comparing the daily rental cost for renting a compact car from Hertz and Avis.
- A random sample of 15 cities is obtained and the following rental information obtained.
- At the .05 significance level can the testing agency conclude that there is a difference in the rental charged?
Example 3 - continued

Data for Hertz

<table>
<thead>
<tr>
<th>City</th>
<th>Hertz</th>
<th>Avis</th>
<th>Avg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miami</td>
<td>43</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>Boston</td>
<td>51</td>
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<td>Chicago</td>
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<tr>
<td>New York</td>
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<tr>
<td>New York</td>
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<tr>
<td>Salt Lake City</td>
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<td>43</td>
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</tr>
<tr>
<td>Seattle</td>
<td>46</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Washington DC</td>
<td>44</td>
<td>43</td>
<td></td>
</tr>
</tbody>
</table>

Data for Avis

<table>
<thead>
<tr>
<th>City</th>
<th>Hertz</th>
<th>Avis</th>
<th>Avg</th>
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</table>

Example 3 - continued

By taking the difference of each pair, variability (measured by standard deviation) is reduced.

\[ \bar{d} = 1.80 \]

\[ s_d = 2.513 \]

\[ n = 15 \]

EXAMPLE 3 continued

\[ H_0: \mu_d = 0 \]

\[ H_1: \mu_d \neq 0 \]

\[ \alpha = 0.05 \]

Matched pairs t test, \( df = 14 \)

\[ H_0 \] is rejected if \( t < -2.145 \) or \( t > 2.145 \)

\[ t = \frac{(1.80) - (2.513)}{\sqrt{15}} = 2.77 \]

Reject \( H_0 \).

There is a difference in mean price for compact cars between Hertz and Avis. Avis has lower mean prices.
Megastat Output – Example 3
Hypothesis Test - Paired Observations

<table>
<thead>
<tr>
<th>Hypothesized Value</th>
<th>Mean Hertz</th>
<th>Mean Hertz (Hertz - Avgs)</th>
<th>Std. Dev.</th>
<th>Std. Error</th>
<th>N</th>
<th>Df</th>
<th>T</th>
<th>P-Value (two-tailed)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>46.667</td>
<td>1.600</td>
<td>2.513</td>
<td>0.649</td>
<td>15</td>
<td>14</td>
<td>2.77</td>
<td>0.0149</td>
</tr>
</tbody>
</table>

Characteristics of F-Distribution

- There is a "family" of F Distributions.
- Each member of the family is determined by two parameters: the numerator degrees of freedom and the denominator degrees of freedom.
- F cannot be negative, and it is a continuous distribution.
- The F distribution is positively skewed.
- Its values range from 0 to $\infty$. As $F \to \infty$, the curve approaches the X-axis.

Test for Equal Variances

- For the two tail test, the test statistic is given by:
  $$F = \frac{S_i^2}{S_j^2}$$
- $S_i^2$ and $S_j^2$ are the sample variances for the two populations.
- There are 2 sets of degrees of freedom: $n_i - 1$ for the numerator, $n_j - 1$ for the denominator.
EXAMPLE 4

A stockbroker at brokerage firm, reported that the mean rate of return on a sample of 10 software stocks was 12.6 percent with a standard deviation of 4.9 percent.

The mean rate of return on a sample of 8 utility stocks was 10.9 percent with a standard deviation of 3.5 percent.

At the .05 significance level, can the broker conclude that there is more variation in the software stocks?

Test Statistic depends on Hypotheses

<table>
<thead>
<tr>
<th>Hypotheses</th>
<th>Test Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_0 : \sigma_1 \geq \sigma_2 )</td>
<td>( F = \frac{s_1^2}{s_2^2} ) use a table</td>
</tr>
<tr>
<td>( H_a : \sigma_1 &lt; \sigma_2 )</td>
<td>( F = \frac{s_2^2}{s_1^2} ) use a table</td>
</tr>
<tr>
<td>( H_0 : \sigma_1 \leq \sigma_2 )</td>
<td>( F = \frac{s_1^2}{s_2^2} ) use a table</td>
</tr>
<tr>
<td>( H_1 : \sigma_1 &gt; \sigma_2 )</td>
<td>( F = \frac{s_1^2}{s_2^2} ) use a table</td>
</tr>
<tr>
<td>( H_0 : \sigma_1 = \sigma_2 )</td>
<td>( F = \frac{\max(s_1^2, s_2^2)}{\min(s_1^2, s_2^2)} ) use a / 2 table</td>
</tr>
<tr>
<td>( H_a : \sigma_1 \neq \sigma_2 )</td>
<td>( F = \frac{\max(s_1^2, s_2^2)}{\min(s_1^2, s_2^2)} ) use a / 2 table</td>
</tr>
</tbody>
</table>

EXAMPLE 4 continued

- \( H_0 : \sigma_1 \leq \sigma_2 \)
- \( H_a : \sigma_1 > \sigma_2 \)
- \( \alpha = 0.05 \)
- F-test
- \( H_0 \) is rejected if \( F > 3.68, \ df = (9, 7) \)
- \( F = \frac{4.9^2}{3.5^2} = 1.96 \) → Fail to Reject \( H_0 \).

There is insufficient evidence to claim more variation in the software stock.
Excel Example

- Using Megastat – Test for equal variances under two population independent samples test and click the box to test for equality of variances
- The default p-value is a two-tailed test, so take one-half reported p-value for one-tailed tests
- Example - Domestic vs Import Data
- $H_0: \sigma_1 = \sigma_2$  $H_a: \sigma_1 \neq \sigma_2$
- $\alpha = .10$
- Reject Ho means use unequal variance t-test
- FTR Ho means use pooled variance t-test

Excel Output

F-test for equality of variance
14.064 variance import
4.654 variance domestic
3.29 F
.0438 p-value
p-value <.10, Reject Ho
Use unequal variance t-test to compare means.

Compare Two Means Flowchart

Two Populations:
- Dependent sampling
- Independent sampling
- Small, $n_1$ and $n_2$ large
- $n_1$ or $n_2$ small
- $G_1 = G_2$
- $G_1 \neq G_2$
- Pooled variance t-test
- Unequal variance t-test