A. The Psychology of Risk Aversion

Suppose a decision maker has an asset worth $100,000 that has a 1% chance of being completely lost. The amount of money the decision maker loses is represented by the discrete random variable $X$ as:

<table>
<thead>
<tr>
<th>X</th>
<th>P(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.99</td>
</tr>
<tr>
<td>-100000</td>
<td>.01</td>
</tr>
</tbody>
</table>

The expected loss is simply the expected value, or mean of this random variable defined as:

$$\mu_X = E(X) = \sum X \cdot P(X) = -1000$$

Therefore, a decision maker who is indifferent to risk would be willing to pay up to $1000 to an insurer to take away this risk. This payment is called a premium.

Suppose the decision maker’s initial wealth was equal to $200,000. Then the decision maker’s wealth taking into account the potential risk is also a discrete random variable $Y$, defined as:

$$Y = 200000 - X$$

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>P(Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200000</td>
<td>.99</td>
</tr>
<tr>
<td>-100000</td>
<td>100000</td>
<td>.01</td>
</tr>
</tbody>
</table>

The decision maker’s expected wealth after taking into account the risk is:

$$\mu_Y = E(Y) = \sum Y \cdot P(Y) = 199000$$

or

$$\mu_Y = 200000 + \mu_X = 199000$$

Now suppose this decision maker was willing to pay $5000 to an insurer to avoid the possibility of losing $100,000. A decision maker who is willing to pay a premium to an insurer that is higher than the expected value of the loss is said to be risk averse. In other words, the decision maker is willing to give up some wealth in order to have security. In this example, the decision maker is willing to accept a reduced wealth of $195,000 to avoid the 1% possibility of a wealth reduced to $100,000.
Now suppose this same decision maker has the same risk as defined above, but instead has initial wealth of $1,000,000. Then this decision maker’s wealth taking into account the potential risk is also a discrete random variable $Y'$, defined as:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y'$</th>
<th>$P(Y')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000000</td>
<td>.99</td>
</tr>
<tr>
<td>-100000</td>
<td>900000</td>
<td>.01</td>
</tr>
</tbody>
</table>

$$
\mu_{Y'} = 1000000 + \mu_X = 999000
$$

This risk averse decision maker may now be willing to pay only $3000 for the insurer to take on this risk, compared to the first scenario. In the first scenario, half of the wealth could be lost, in this scenario only 10% of the wealth is at risk. This example shows that risk aversion is dependent upon total initial wealth.

In general, $G$ is the total gross premium a decision maker with initial wealth $w$ is willing to pay to an insurer to take on a risk that has an expected loss of $\mu_X$, where $X$ is a random variable. Then, a decision maker is risk averse if:

$$
G > \mu_X
$$

Insurance is only feasible when a decision maker is willing to pay a premium greater than the expected value of the loss, since the insurance company has a need to load the pure premium for expenses and profits, and may be risk averse itself.

**B. Utility Theory**

From above, we see that the risk a decision maker is willing to take is dependent upon wealth. It is also true that the premium the decision maker is willing to pay ($G$) is greater than the expected value of the loss ($\mu$). This philosophy of risk aversion can be described by a utility function that is dependent on wealth, which will be represented by $u(w)$. In the first example above, the decision maker with wealth of $200,000$ was willing to pay up to $5000$ for insurance against a 1% possibility of losing $100,000$. However, with wealth of $1,000,000$, the decision maker may only pay $3,000$ for the same insurance. We can describe the relationship of complete insurance as:

$$
u(w-G) = E(u(w))$$

Then by substituting both decision maker’s data into the equation:

$$
u(195000) = .01*u(100000) + .99*u(200000)$$
$$
u(997000) = .01*u(900000) + .99*u(1000000)$$

Note that there are many utility functions that will solve these equations. Also any linear transformation $au(w) + b$ will have the same utility as the original utility function.
Two properties of a risk averse utility function are that it must be increasing and concave. In other words the first and second derivatives must be defined and as follows:

\[ u'(w) > 0 \text{ and } u''(w) < 0 \]

This also leads to **Jensen’s Inequality** which states that if \( X \) is a random variable and \( u'(X)>0 \) and \( u''(X)<0 \), then

\[ u(E(X)) \geq E(u(X)) \]

These properties lead to a commonly used measure of risk aversion obtained from the utility function: the **absolute risk aversion**, \( R_A(w) \) which contains useful information about the decision maker’s attitude towards risk by measuring the relative concavity of the utility function \( u(w) \). For risk averse decision makers, \( R_A(w) \) will be positive:

\[ R_A(w) = -\frac{u''(w)}{u'(w)} \]

Finally, the decision to purchase insurance or to assume a risk denoted by random variable \( X \) can now be compared using the utility function:

\[ u(w-G) = E[u(w-X)] \]

\( G \) is the maximum premium that a decision maker will pay for insurance.

**C. TYPES OF UTILITY FUNCTIONS**

The **Exponential** utility function is used extensively in finance and insurance applications. The function, its first two derivatives and absolute risk aversion are:

\[ u(w) = 1 - e^{-aw} ; a > 0 \]
\[ u'(w) = ae^{-aw} \]
\[ u''(w) = -a^2e^{-aw} \]
\[ R_A(w) = a \]

This family of exponential functions qualify as risk averse utility functions, and the absolute risk aversion is a constant, in other words the maximum premium a decision maker will pay for an insurance does not depend on \( w \). Additional, finding the expected value of the utility function is essentially the same as finding the moment generating function of \( w \):

\[ E(1 - e^{-aw}) = 1 - E(e^{-aw}) = 1 - M_w(-a) \]
Example:

A decision maker’s utility function is given by \( u(w) = 1 - e^{-5w} \). There are two investment options available to this decision maker, the first choice is Normal with \( \mu = 5 \), and \( \sigma^2 = 2 \). The second choice is Normal with \( \mu = 6 \) and \( \sigma^2 = 2.5 \). The moment generating function for the Normal Distribution is known to be \( M_w(t) = \exp(-\mu t + t^2 \sigma^2/2) \) Which choice will this decision maker prefer?

**Choice 1:**
\[
E(u(w)) = 1 - M_w(-5) \\
= 1 - \exp\{-5(5) + (5^2)(2)/(2)\} \\
= 0
\]

**Choice 2:**
\[
E(u(w)) = 1 - M_w(-5) \\
= 1 - \exp\{-5(6) + (5^2)(2.5)/(2)\} \\
= 1 - e^{1.25}
\]

Since Choice 1 has a higher expected utility, it is the preferred option for this decision maker.

The **Power** utility function is defined as follows:

\[
u(w) = \left(\frac{wa - 1}{a}\right) ; \ 0 < a < 1
\]

\[
u'(w) = wa^{-1}
\]

\[
u''(w) = (a - 1)wa^{-2}
\]

\[R_A(w) = (1 - a)/w\]

The advantage of the power utility function is that absolute risk aversion will decrease as wealth increases. This is consistent with the first example where the wealthier individual was less inclined to pay a large premium for insurance.

Example:

A decision maker’s utility function is defined as \( u(w) = \sqrt{w} \). The decision maker has wealth = 10 and faces a random loss \( X \) with uniform distribution on (0,10). What is the maximum amount this decision maker will pay for complete insurance against this loss?

\[
u(w - G) = E(u(w - X))
\]

\[
\sqrt{10 - G} = E(\sqrt{10 - X})
\]

\[
\sqrt{10 - G} = \int_0^{10} (\sqrt{10 - X})/10 \, dx
\]

\[
\sqrt{10 - G} = (2/3)\sqrt{10}
\]

\[
G = 5.5556
\]
The **Quadratic** utility function is defined as:

\[
\begin{align*}
  u(w) &= w - w^2/2b ; \ w < b \\
  u'(w) &= 1 - w/b \\
  u''(w) &= -1/b \\
  R_A(w) &= 1/(b-w)
\end{align*}
\]

Although this form of utility function is convenient in that it depends only on the first two moments of a random variable, it has the undesirable property of increased risk aversion as wealth increases, which is contrary to the psychology of risk aversion.

**Example:**

A decision maker’s utility function is given by \( u(w) = w - .01w^2 ; \ w < 50 \). Find the maximum premium that the decision maker will pay for the following conditions: wealth = 10, potential loss = 10, probability of loss = .5. This reduces to solving a quadratic equation:

\[
\begin{align*}
  u(w-G) &= E(u(w-X)) \\
  u(10-G) &= .5u(10) + .5u(10-10) \\
  10 - G - .01(10 - G)^2 &= .5(10 - .01(10^2)) \\
  G &= 5.28
\end{align*}
\]

The **Logarithmic** utility function is defined as:

\[
\begin{align*}
  u(w) &= a \log(w) + b ; \ a > 0 \\
  u'(w) &= a / w \\
  u''(w) &= -a / w^2 \\
  R_A(w) &= 1 / w
\end{align*}
\]

Like the Power utility function, the absolute risk aversion will decrease as wealth increases.

**Example:**

A decision maker’s utility function is defined as \( u(w) = \log(w) \). The decision maker has wealth = 10 and faces a random loss 5 with probability of .25. What is the maximum amount this decision maker will pay for complete insurance against this loss?

\[
\begin{align*}
  u(w-G) &= E(u(w-X)) \\
  \log(10-G) &= (.75)\log(10) + (.25)\log(5) \\
  \log(10-G) &= .924743 \\
  10 - G &= 10^{.924743} \\
  G &= 1.591
\end{align*}
\]
Homework Problems

1. A decision maker’s utility function is given by \( u(w) = 1 - e^{-3w} \). There are two investment options available to this decision maker, the first choice is Normal with \( \mu = 6 \), \( \sigma^2 = 2 \). The second choice is Normal with \( \mu = 8 \) and \( \sigma^2 = 3 \). The moment generating function for the Normal Distribution is known to be \( M_w(t) = \exp(\mu t + t^2 \sigma^2/2) \). Which choice will this decision maker prefer?

2. A decision maker’s utility function is defined as \( u(w) = \sqrt{w} \). The decision maker has wealth = 10 and faces a random loss \( X \) of 0, 1 or 2, each with a probability of 1/3. What is the maximum amount this decision maker will pay for complete insurance against this loss? Show that this choice is risk averse, that is, show the maximum premium is greater than the expected loss.

3. An decision maker and an insurance company both are risk averse with utility function \( u(w) = \log(w) \). The decision maker has wealth of 20 and faces a loss of 10 with probability .10. The insurance company has wealth of 100. What is the maximum premium the decision maker will spend for complete insurance? What is the minimum premium the insurance company will accept for complete insurance?