Handout 2 discussed the short-term risk model, which uses random variables dealing with claim probability and claim amount. In the short-term risk model, it is sufficient to think of survival as a yes/no question, and to use the Indicator or Bernoulli random variable for this model.

The next step is to add a random variable to represent the time until the claim occurs. These random variables are referred to as survival or time until termination or failure distributions. In theory, survival distributions can be modeled by parametric distributions. In the past, this also had much practical application in actuarial work, however today with cheaper, faster computers, empirical distributions formed from experience and census data are the standard. The most common example of an empirical survival distribution is a life or mortality table.

Some Survival Model Concepts

The random variable \( T \) will represent the time until “failure.” The **Cumulative Distribution Function** \( F(t) = \Pr(T \leq t) \) represents the probability failure occurs before time \( t \) and the **Survival Distribution Function** \( S(t) = \Pr(T > t) \) represents the probability of surviving past time \( t \). It is clear that \( F(t) = 1 - S(t) \).

Actuarial terminology is to use \( p \) to represent survival and \( q \) to represent failure. In particular, \( p_0 \) represents the probability an infant at birth survives to time \( t \) and \( q_0 \) represents the probability death occurs before time \( t \):

\[
, \ p_0 = S(t) \quad , q_0 = F(t) \quad , p_0 + , q_0 = 1
\]

In general, \( p_x \) represents the probability an individual age \( x \) survives another \( t \) years and \( q_x \) represents death of the same individual before \( t \) has elapsed. Using conditional probability and survival model terminology:

\[
, p_x = \frac{S(x + t)}{S(x)} \quad , q_x = 1 - \frac{S(x + t)}{S(x)}
\]

**Example 1:** Suppose survival follows a uniform distribution \( S(t) = (100-t)/100 \) over \((0,100)\). Find the probability of an individual age 60 surviving (or not surviving) 5 more years.

\[
, p_{60} = \frac{S(65)}{S(60)} = \frac{35/100}{40/100} = .875 \quad , q_{60} = 1-.875 = .125
\]
The **Probability Density Function**, \( f(t) \) is the derivative of \( F(t) \).

\[
f(t) = F'(t) = -S'(t)
\]

So the Cumulative and Survival Distribution functions are:

\[
F(t) = \int_0^t f(x) \, dx \quad S(t) = \int_t^\infty f(x) \, dx
\]

This density function is the unconditional density of failure at time \( t \). If we condition this density upon survival to time \( t \), we have an instantaneous measure of failure at time \( t \). In survival models, this is called the **hazard rate** and represented by the function \( \lambda(t) \). In actuarial models, the hazard rate is referred to as the **force of mortality** \( (\mu_t) \) at time \( t \):

\[
\lambda(t) = \mu_t = \frac{f(t)}{S(t)}
\]

**Example 2:** Using the uniform survival model from Example 1, find the force of mortality at age 50 and age 75:

\[
S(t) = (100-t)/100 \quad f(t) = (1/100)
\]

\[
\lambda(t) = \frac{(1/100)}{(100-t)/100} = \frac{1}{100-t}
\]

\[
\mu_{50} = \lambda(50) = .02 \quad \mu_{75} = \lambda(75) = .04
\]

Since \( f(t) = -S'(t) \) we can write \( S(t) \) in terms of the hazard rate:

\[
\lambda(t) = \frac{-S'(t)}{S(t)} = -\frac{d}{dt} \ln S(t)
\]

\[
\int_0^t \lambda(x) \, dx = -\ln S(t)
\]

\[
S(t) = \exp \left[ -\int_0^t \lambda(x) \, dx \right]
\]

**Example 3:** Find the survival model based on a constant hazard rate \( \lambda(t) = \lambda \)

\[
S(t) = \exp \left[ -\int_0^t \lambda(x) \, dx \right] = e^{-\lambda t}
\]

or the exponential distribution. A constant hazard rate that does not depend on \( t \) is consistent with the “memoryless” feature of the exponential distribution.
The unconditional expectation of survival is the expected value of T in survival model theory and referred to as the complete expectation of life \( e_t \) in actuarial science. \( E(T^2) \) and variance can be defined in the usual way:

\[
E(T) = e_t = \int_0^\infty t f(t) dt \\
E(T^2) = \int_0^\infty t^2 f(t) dt \\
VAR(T) = E(T^2) - (E(T))^2
\]

**Example 4:** Find the complete expectation of life and variance for the uniform model in Example 1:

\[
e_t = \int_0^{100} t (1/100) dt = 50 \\
E(T^2) = \int_0^{100} t^2 (1/100) dt = 3333.33 \\
VAR(T) = 833.33
\]

These concepts will be developed further when we introduce the actuarial table. The following table summarizes the difference in terminology between actuarial and survival theory:
<table>
<thead>
<tr>
<th>Concept</th>
<th>Survival Theory</th>
<th>Actuarial Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability of Survival from age 0 to x</td>
<td>( S(x) )</td>
<td>( x P_0 )</td>
</tr>
<tr>
<td>Probability of death no later than age x</td>
<td>( F(x) )</td>
<td>( x q_0 )</td>
</tr>
<tr>
<td>Conditional probability of survival from age x to x+n</td>
<td>( \frac{S(x+n)}{S(x)} )</td>
<td>( n P_x )</td>
</tr>
<tr>
<td>Conditional probability of death before age x+n, given alive at age x</td>
<td>( \frac{S(x) - S(x+n)}{S(x)} )</td>
<td>( n q_x )</td>
</tr>
<tr>
<td>Hazard rate (force of mortality) at exact age x</td>
<td>( \lambda(x) = -\frac{S'(x)}{S(x)} = -\frac{d}{dx} \ln S(x) )</td>
<td>( \mu_x )</td>
</tr>
<tr>
<td>Density function for death at exact age x</td>
<td>( f(x) = F'(x) = -S'(x) )</td>
<td>( x P_0 \mu_x )</td>
</tr>
<tr>
<td>Expectation of future lifetime at birth</td>
<td>( E(X) = \int_0^\infty x f(x) dx )</td>
<td>( e_x = \int_0^\infty (x)_x p_0 \mu_x dx )</td>
</tr>
<tr>
<td>Variance of future lifetime at birth</td>
<td>( Var(X) = \int_0^\infty x^2 f(x) dx - [E(X)]^2 )</td>
<td>No specific symbol</td>
</tr>
</tbody>
</table>