Stat 6863-Handout 5  
Fundamentals of Interest  
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The prior handouts addressed benefit claim occurrences, amounts, total claims and time-until-termination as random variables. The final component of the actuarial model involves the economic assumptions such as rate of return on assets and inflation. This brief overview will introduce the fundamental principles of the deterministic approach to the model, which is still the most widely used approach in actuarial science.

Interest Theory Definitions:

The most basic model is the assumption of a constant rate of interest over time. This is a special case of the deterministic model.

The **principal** is the initial amount of money invested and the amount received after a period of time is called the **accumulated value**. The difference of these two amounts is called **interest**.

An accumulation function, $a(t)$ gives the value of an initial principal of 1 at some future time $t$. If the initial principal was some value $k$, then the accumulated value at time $t$ is written as $k \cdot a(t)$ and the interest earned is $k \cdot [a(t) - 1]$.

The effective rate of interest, $i$, is the amount that 1 unit will earn during one year of investment: $i = a(1) - a(0)$ or $a(1) = 1 + i$.

There are two basic methods of crediting interest over different periods of time: **Simple interest** is an arithmetic accumulation of interest over time: $a(t) = 1 + i \cdot t$ while **Compound interest** is a geometric accumulation of interest over time: $a(t) = (1 + i)^t$. In most cases, compound interest is the preferred method of calculation.

**Example:** An initial investment of $1000 earns 6% interest for 5 years. Find the accumulated value under both methods of crediting interest.

$$
\begin{align*}
\text{Simple} & : 1000[1 + (.06)(5)] = 1300.00 \\
\text{Compound} & : 1000(1.06)^5 = 1338.23
\end{align*}
$$

The term $1+i$ is frequently referred to as an **accumulation factor** as it calculates a value at the end of a year of an initial investment at the beginning of the year. However, we quite often do the reverse, to find the current value of a payment in the future. The **discount factor**, $v = 1/(1+i)$, is the value at the beginning of the year of 1 unit at the end of the year.

The reciprocal of the accumulation function is called the **discount function** and can be written for both simple and compound interest:

$$
\begin{align*}
\text{Simple} & : a^{-1}(t) = 1/(1 + it) \\
\text{Compound} & : a^{-1}(t) = 1/(1 + i)^t = v^t
\end{align*}
$$
In particular \( v^t \) is called the **present value** of 1 unit to be paid after \( t \) years.

**Example:** A zero coupon bond with a face value of $100,000 will be payable in 10.5 years. Find the present value of the bond assuming a 7% rate of interest.

\[
PV = 100000 \cdot (1/1.07)^{10.5} = 49144
\]

An effective rate of discount, or **discount rate** (represented by the letter \( d \)), is the measure of interest paid at the beginning of the year, as opposed to the effective rate of interest which is credited at the end of the year.

There are several relationships between \( d \), \( v \) and \( i \):

\[
i = \frac{d}{1-d} \quad d = \frac{i}{1+i} = 1 - v = iv \quad v' = (1-d)'
\]

**Example:** A bank lends a borrower $1000 and immediately collects $80 of interest, leaving the borrower with $920. Then \( d = 80/1000 = 8\% \) is the discount rate. The effective rate of interest is \(.08/(1-.08) = .08696\)

The effective rates of interest and discount assume payment of interest at the end and beginning of the year respectively. However, interest may be paid more frequently during a year, for example monthly interest on credit cards or daily interest on savings accounts. An annual rate of interest that is paid in units of \( 1/m \) over \( m \) equal time segments during the year is called the **nominal rate of interest** and is denoted by the symbol \( i^{(m)} \). Assuming compound interest, the following formulas apply:

\[
i^{(m)} = m[(1+i)^{1/m} - 1] \quad i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1
\]

**Example:** A credit card charges a nominal rate of interest of 12\% compounded monthly. Determine the effective rate of interest.

\[
i^{(12)} = .12 \quad i = \left(1 + \frac{.12}{12}\right)^{12} - 1 = 12.68\%
\]

The **force of interest**, \( \delta \), is an annual rate of interest that is paid continuously over the year. In other words, it is the instantaneous rate of interest:

\[
\delta = \lim_{m \rightarrow \infty} i^{(m)} = \ln(1+i) \quad i = e^\delta - 1
\]

also \( d < d^{(2)} < d^{(3)} < ... < \delta < ... < i^{(3)} < i^{(2)} < i \) for \( i > 0 \)

**Example:** Find the force of interest when the effective rate of interest is 8\%. Find the effective rate of interest when the force of interest is 12\%.
\[ \delta = \ln(1.08) = .0769 \quad i = e^{12} - 1 = .1275 \]

A series of payments in equal installments is called an **annuity**. An **annuity-immediate** is when the payments are made annually at the end of the period. The present value of an n-year annuity-immediate of annual installments of one unit is written as \( a_n \). An **annuity-due** is when the payments are made annually at the beginning of the period. The present value of an n-year annuity-due of annual installments of one unit is written as \( \ddot{a}_n \).

\[
\begin{align*}
    a_n &= v + v^2 + \cdots + v^n = \frac{1-v^n}{i} \\
    \ddot{a}_n &= 1 + v + v^2 + \cdots + v^{n-1} = 1 + a_{n-1} = (1+i)a_n
\end{align*}
\]

**Example:** Find the present value of 10 equal payments of $1000 at the end of the year assuming an interest rate of 10%:

\[
PV = 1000a_{10} = 1000 \left( \frac{1-(1/1.1)^{10}}{0.1} \right) = 6144.57
\]

**Example:** Pension law requires that 401(k) Plan distributions made to a participant before age 59.5 are subject to a early distribution penalty of 10%. One option to avoid this penalty is to make “substantially equivalent” annual payments over the life expectancy of the participant. A participant currently age 54 has an account of $200,000 and a life expectancy of 25 years. Determine the annual amount payable at the beginning of the year that would satisfy this requirement assuming the interest rate is 9% per year.

\[
Pmt = \frac{200000}{\ddot{a}_{25}} = \frac{200000}{\left( \frac{1 - (1/1.09)^{25}}{0.09} \right)} = \frac{200000}{10.8048} = 18510
\]

The accumulated value of an annuity-immediate which pays one unit at the end of the year for n years is called the **future value** and is represented by the symbol \( s_n \). The future value for an annuity-due is represented by \( \ddot{s}_n \):

\[
\begin{align*}
    s_n &= 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} = \frac{(1+i)^n - 1}{i} = a_n (1+i)^n \\
    \ddot{s}_n &= (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} + (1+i)^n = (1+i) \left( \frac{(1+i)^n - 1}{i} \right) = \ddot{a}_n (1+i)^n
\end{align*}
\]

**Example:** An investor puts $2000 into savings at the end of each of 10 years. Find the value of the account after 10 years assuming a 7% annual rate of return.
\[ FV = 2000s_{10|0} = (2000) \frac{1.07^{10} - 1}{.07} = 27633 \]

In practice, annuities are usually paid more frequently than annually. The present value of an annuity which pays 1/m at the end of each mth of the year for a total of n years is denoted by \( a^{(m)}_{\overline{n|m}} \). The present value of an annuity which pays 1/m at the beginning of each mth of a year for a total of n years is \( \ddot{a}^{(m)}_{\overline{n|m}} \):

\[
a^{(m)}_{\overline{n|m}} = \frac{1}{m} \left[ \frac{1}{v^m} + \frac{2}{v^{2m}} + \cdots + \frac{n-1}{v^{(n-1)m}} + v^n \right] = \frac{1 - v^n}{i^{(m)}}
\]

\[
\ddot{a}^{(m)}_{\overline{n|m}} = \frac{1}{m} \left[ \frac{1}{1 + v^m} + \frac{2}{v^{2m}} + \cdots + \frac{n-1}{v^{(n-1)m}} \right] = \left( 1 + \frac{i^{(m)}}{m} \right) \frac{1 - v^n}{i^{(m)}} = \left[ 1 + \frac{i^{(m)}}{i^{(m)}} \right] \frac{1 - v^n}{i^{(m)}}
\]

**Example:** A pension plan is paying a 10 year certain benefit of $1000 monthly for 120 consecutive months. Find the present value of this benefit assuming benefits are paid at the beginning of the month and interest of 8% per year.

\[
\ddot{a}^{(12)}_{\overline{120|12}} = 1.08^{12} \left[ \frac{1 - (1/1.08)^{10}}{12 \left( 1.08^{12} - 1 \right)} \right] = 6.9974 \quad \text{PVBA} = (12000)(6.9974) = 83790
\]

A similar adjustment can be made for future value which will be omitted from this handout.

When we let the frequency of the payment becomes infinite, we get the theoretical **continuous annuity** \( \ddot{a}_{\overline{n|m}} \), which can also be used to approximate annuities of great frequency (e.g. daily):

\[
\ddot{a}_{\overline{n|m}} = \lim_{m \to \infty} \ddot{a}^{(m)}_{\overline{n|m}} = \lim_{m \to \infty} a^{(m)}_{\overline{n|m}} = \int_0^n v^d t = \frac{1 - v^n}{\delta} = \frac{i}{\delta} a_{\overline{n|m}}
\]

**Example:** Annual payments of $1000 are paid continuously over the next 5 years. Assuming an effective interest rate of 6% per year, determine the present value.

\[
(1000) \ddot{a}_{\overline{5|5}} = (1000) \left[ \frac{1 - v^n}{\delta} \right] = (1000) \left[ \frac{1 - (1.06)^{-5}}{\ln(1.06)} \right] = $4337.51
\]

Also \( a_{\overline{n|m}} < a^{(2)}_{\overline{n|m}} < a^{(3)}_{\overline{n|m}} < \cdots < \ddot{a}_{\overline{n|m}} \), \( \ddot{a}^{(2)}_{\overline{n|m}} < \ddot{a}^{(3)}_{\overline{n|m}} < \ddot{a}_{\overline{n|m}} \) for \( i > 0 \)
There are many other financial functions that can be derived, but will be omitted here. In practice, these calculations can be easily performed using the financial functions in Excel. One application is the relationship between the Present (or Future) value, the interest rate, the number of payments and the amount of payment. Once three of these items are known, the other can be derived. These functions in Excel are called PV (or FV), RATE, NPER, and PMT.

In a later handout, we will discuss interest as a random variable which reflects the diversity of modern investment theory.

Homework:

1. For an effective interest rate of 6.5% derive the following terms:

   \[ d \quad v^6 \quad \delta \quad \bar{a}_\overline{n|} \quad s_\overline{n|} \quad i^{(12)} \quad a_\overline{n|}^{(12)} \quad \bar{a}_\overline{n|} \]

2. Each year, a corporation makes 4 quarterly contributions of $1,000,000 to its pension plan trust. These payments occur on March 31, June 30, September 30 and December 31. Assuming no initial value and an interest rate of 8.5%, what is the value of the pension trust after 10 years of contributions?

3. You purchase a home for $500,000 with a 20% down payment and a 30 year fixed-rate mortgage. The terms of this mortgage are payments at the end of the month and a nominal interest rate of 8% per year charged monthly. Calculate the monthly mortgage payment. What is the effective rate of annual interest for this transaction?

4. After 5 years, you refinance the mortgage described in Question 3 and switch to a new 15 year fixed-rate mortgage at a nominal rate of 5.5% charged monthly. Determine the new mortgage payment.

5. A bond with a face value of $100,000 will pay semi-annual coupons of $2000 at the end of each six month period. At the end of 8 years, the bond will mature and return $100,000 to the holder, in addition to the final $2000 coupon. Determine the present value of the bond under the following annual interest rate scenarios: 3%, 5%, 7%.

6. An annuity payable at the beginning of each month pays $2000 per month for the first 5 years and then $1000 per month for the next 5 years. Calculate the present value of this annuity using an 8% effective rate of interest.

7. Repeat question 5 assuming the payments are continuous through the year.