The following method has replaced commutation functions in construction of life table insurances and annuities.

**Recursion Formulas for Insurances and Annuities:**

Rather than using commutation functions, most life tables are built today using a recursive process as follows:

- Define $A_x$ and $\ddot{a}_x$ for the last age on the table $(\omega - 1)$.
- Build backwards by recursion to complete the table
- Other types of annuities and insurances can be developed from these two values.

Formulas for a Whole Life Insurance: $A_{\omega-1} = v A_x = vq_x + vp_x A_{x+1}$
Formulas for a Life Annuity Due: $\ddot{a}_{\omega-1} = 1 \ddot{a}_x = 1 + vp_x \ddot{a}_{x+1}$

Example: Complete the Mortality Table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$qx$</th>
<th>$lx$</th>
<th>$dx$</th>
<th>$vx(8%)$</th>
<th>$1000A_x$</th>
<th>$\ddot{a}_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.25</td>
<td>100000</td>
<td>25000</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>75000</td>
<td>15000</td>
<td>0.9259259</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>60000</td>
<td>18000</td>
<td>0.8573388</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>42000</td>
<td>21000</td>
<td>0.7938322</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>21000</td>
<td>21000</td>
<td>0.7350299</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

First build the bottom row:

$A_4 = (1.08)^{-1} = .9259259 \quad \ddot{a}_4 = 1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$qx$</th>
<th>$lx$</th>
<th>$dx$</th>
<th>$vx(8%)$</th>
<th>$1000A_x$</th>
<th>$\ddot{a}_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.25</td>
<td>100000</td>
<td>25000</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>75000</td>
<td>15000</td>
<td>0.9259259</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>60000</td>
<td>18000</td>
<td>0.8573388</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>42000</td>
<td>21000</td>
<td>0.7938322</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>21000</td>
<td>21000</td>
<td>0.7350299</td>
<td>925.92593</td>
<td>1.000</td>
</tr>
</tbody>
</table>
Then build backwards up to row 0

\[
A_3 = 0.5 \cdot (1.08)^{-1} + 0.9259259 \cdot 0.5 \cdot (1.08)^{-1} = 0.89163237
\]

\[
\ddot{a}_3 = 1 + 0.5 \cdot (1.08)^{-1} \cdot 1.000 = 1.463
\]

etc.

<table>
<thead>
<tr>
<th>x</th>
<th>qx</th>
<th>lx</th>
<th>dx</th>
<th>vx(8%)</th>
<th>1000A_x</th>
<th>\ddot{a}_x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.25</td>
<td>100000</td>
<td>25000</td>
<td>1</td>
<td>800.25085</td>
<td>2.697</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>75000</td>
<td>15000</td>
<td>0.9259259</td>
<td>819.02789</td>
<td>2.443</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>60000</td>
<td>18000</td>
<td>0.8573388</td>
<td>855.68765</td>
<td>1.948</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>42000</td>
<td>21000</td>
<td>0.7938322</td>
<td>891.63237</td>
<td>1.463</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>21000</td>
<td>21000</td>
<td>0.7350299</td>
<td>925.92593</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Formulas for determining other types of insurances and annuities:

Term Insurance:

\[
A^1_{x\bar{n}} = A_x - n E_x \cdot A_{x+n} \text{ (recall } n E_x = v^x_n P_x )
\]

Insurance payable at death:

\[
\overline{A}_x = A_x \left( \frac{i}{\delta} \right)
\]

Mthly annuity due:

\[
\ddot{a}^{(m)}_x \approx \ddot{a}_x - \left( m - 1 \right)/2m
\]

The following annuity formulas are valid for any type; mthly annuities are shown:

Deferred life annuity:

\[
\dddot{a}^{(m)}_x = n E_x \cdot \ddot{a}^{(m)}_{x+n}
\]

Temporary life annuity:

\[
\dddot{a}^{(m)}_{x\bar{n}} = \ddot{a}^{(m)}_x - n \ddot{a}^{(m)}_{x+n}
\]

Certain and Life annuity:

\[
\dddot{a}_{x\bar{n}} = \dddot{a}^{(m)}_{x\bar{n}} + n \ddot{a}^{(m)}_{x+n}
\]

Some Examples:

\[
A^1_{2,0\bar{1}} = A_2 - E_2 \cdot A_3 = 0.85568765 \cdot (1.08)^{-1} \cdot (42000 / 60000) \cdot 0.7938322 = 0.2777778
\]

\[
\overline{A}_3 = A_3 \left( \frac{i}{\delta} \right) = 0.8556876(0.08/\ln(1.08)) = 0.9268402
\]

\[
\ddot{a}_2^{(12)} \approx \ddot{a}_2 - (11)/24 = 1.948 - 11/24 = 1.490
\]

\[
\ddot{a}_3^{(12)} = \ddot{a}_1^{(12)} - E_1 \cdot \ddot{a}_3^{(12)} = (2.443 - 11/24) - (1.08)^{-2} (42000 / 75000) (1.463 - 11/24) = 1.502
\]
Variance of Life Insurance:

Recall that $A_x = E[Z_T]$ where $Z = \text{present value of }$ $1$ payable upon death of $x$ at future age $x+t$. This means $A_x = E[v^x] = E[e^{-\delta t}] = \int_0^\infty e^{-\delta t} p_x(t) dt$.

Finding any moment of $Z$ is simply a matter of changing the force of interest, since $Z^j = (e^{-\delta t})^j = (e^{-(j\delta)})^j$. In particular, $A_x = E[Z^2] = E[Z]$ evaluated at force of interest $2\delta$.

The variance of $A_x$ can then be easily calculated as $Var(A_x) = E[A_x] - (E[A_x])^2$.

Example: Find the Variance of a whole life policy issued to age 30 when mortality is modeled by pdf Uniform $(0,100)$. Assume a force of mortality of 5%.

$A_{30} = \int_0^{70} e^{-0.05t} \frac{1}{100}(1-e^{-3.5}) dt = .27709$

$A_{30}^2 = \int_0^{70} e^{-0.10t} \frac{1}{100}(1-e^{-7}) dt = .14273$

$Var(A_{30}) = .14273 - (.27709)^2 = .06595$

If 100 identical policies of $1000 each as described above are sold by an insurer, determine the net premium loading so that the insurer would be 95% sure of having adequate reserves to pay all claims.

Let $S = \text{Sum of all claims paid}$.

$E(S) = (100)(1000)(.27709) = 27709$

$Var(S) = (100)(1000)^2(.06595) = 6595000$

$SDev(S) = 2568$

$95^{\text{th}}$ Percentile of $S \approx 27709 + (1.645)(2568) = 31933$

$\theta = 31933 / 27709 - 1 = 15.2\%$ loading of net premium.