Review of Insurance

In prior handouts we have addressed the calculation of Net Single Premiums for different types of insurance, most notably whole life and term insurance. In these handouts we developed the formulas for determining the net single premium for both the whole life and term life insurance policies payable at the moment of death:

\[
\overline{A}_x = E(Z) = \int_0^\infty v^t p_x \mu_{x+t} dt \\
\overline{A}_{x|n} = E(Z) = \int_0^n v^t p_x \mu_{x+t} dt
\]

There are analogous discrete Net Single Premiums for insurances payable at the end of the year death occurs that can be derived from annuity functions:

\[
A_x = \sum_{k=0}^{\infty} v^{k+1} p_x q_{x+k} = 1 - d \cdot \overline{a}_x \\
A_{x|n} = \sum_{k=0}^{n-1} v^{k+1} p_x q_{x+k} = A_x - n E_x \cdot A_{x+n}
\]

Since most insurances are paid at the moment of death, the following formulas will adjust these easier to derive discrete insurance premiums to the continuous premiums payable at the moment of death (Under UDD):

\[
\overline{A}_x = \frac{i}{\delta} A_x \\
\overline{A}_{x|n} = \frac{i}{\delta} A_{x|n}
\]

Where \( \delta = \ln(1+i) \)

The net single premiums would have the same actuarial present value as one unit payable immediately upon death under the whole life example, and upon death before \( n \) years under the term-life example.

Premiums

In practice, insurance contracts are not funded by a single premium, but by periodic premiums that would end when the insurance contract terminated. For example, a whole life policy may have equal semi-annual payments that do not increase over time. We need a method of determining these premiums.

Let’s first analyze the fully continuous problem of a Whole Life insurance sold to age \( x \) that is payable at the moment of death \( T \), and will be funded by a level annual premium \( \overline{P} (\overline{A}_x) \) that will be paid continuously until \( T \). Let \( L \) be the random variable that is the difference between the present values of the Insurance paid and the collected premium. Then \( L \) can be characterized as \( L(T) = v^T - \overline{P} \overline{a}_{\overline{r}} \) and the expected value of \( L(T) \) is
calculated from prior work involving expected values of insurances and annuities:

\[ E(L(T)) = \int_0^\infty v', p_x, \mu_x dt - \overline{P}(\overline{A}_x) \int_0^\infty v', p_x dt \]

\[ = \overline{A}_x - \overline{P}(\overline{A}_x) \overline{a}_x \]

To determine a net premium, set \( E(L(T)) = 0 \) and solve for the annual level premium for the whole life policy and analogously for the term life policy:

\[ \overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x} \quad \overline{P}(\overline{A}_{x|n}) = \frac{A_{x|n}^1}{a_{x|n}^{(m)}} \]

Typically, insurance companies will collect premiums annually, semi-annually, quarterly or monthly. Fully and partially discrete premiums that are payable monthly during the period have very similar formulas:

\[ P^{(m)}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x^{(m)}} \quad P^{(m)}(\overline{A}_{x|n}) = \frac{A_{x|n}^1}{a_{x|n}^{(m)}} \]

Another funding option is the collect the premium over a limited amount of time (h years) so the insurance is fully funded at age x+h. For the n-year term life policy, we assume \( h \leq n \). The formulas above would then be modified by changing the life annuity to a temporary annuity:

\[ hP^{(m)}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x^{(m)\times h}} \quad hP^{(m)}(\overline{A}_{x|n}) = \frac{A_{x|n}^1}{a_{x|n}^{(m)\times h}} \]

\[ hP^{(m)}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x^{(m)\times h}} \quad hP^{(m)}(\overline{A}_{x|n}) = \frac{A_{x|n}^1}{a_{x|n}^{(m)\times h}} \]

**Example 1:** Determine the annual level premium \( \pi \) for a $100,000 whole life policy payable at the moment of death for a 40 year old male, interest rate =8% and under GAM94M.

\[ \overline{a}_x = 12.44309 \]

\[ A_x = 1 - (0.08/1.08) \times 12.44309 = 0.078289 \]

\[ \overline{A}_x = 0.078289 (0.08/\ln 1.08) = 0.081381 \]

\[ \pi = 100000P(\overline{A}_x) = (100000)(0.081381)/(12.44309) = 654.02 \]
**Example 2:** Determine the semi-annual level premium $\pi$ for a $200,000 whole life policy payable at the moment of death for a 50 year old male, interest rate =8% and under GAM94M.

\[
\bar{a}_x = 11.52533 \\
A_x = 1 - (.08/1.08) \times 11.52533 = .146272 \\
\overline{A}_x = .146272 \times (\frac{.08}{\ln 1.08}) = .152048 \\
\hat{a}^{(2)}_{x:30} = 9.94592 \text{ (see annuity handout for methodology)} \\
\pi = (1/2)(200000) \times P^{(2)}(\overline{A}_x) = (1/2)(200000)(.152048)/(9.94592) = \$1528.75
\]

**Example 3:** Determine the semi-annual level premium $\pi$ for a $250,000 10 year term-life policy payable at the moment of death for a 45 year old male, interest rate =8% and under GAM94M.

\[
\bar{a}_x = 12.05057 \\
\bar{a}_{x+10} = 10.84802 \\
E_x = .4490938 \\
A_x = 1 - (.08/1.08) \times 12.05057 = .107365 \\
A_{x+10} = 1 - (.08/1.08) \times 10.84802 = .196443 \\
A^1_{x:10} = .107365 - (.4490938)(.196443) = .0191439 \\
\overline{A}^1_{x:10} = .0191439 \times (\frac{.08}{\ln 1.08}) = .0198998 \\
\hat{a}^{(2)}_{x:10} = 7.04106 \text{ (see annuity handout for methodology)} \\
\pi = (1/2)(250000) \times P^{(2)}(\overline{A}^1_{x:10}) = (1/2)(250000)(.0198998)/(7.04106) = \$353.28
\]

**Benefit Reserves**

All net premiums are calculated assuming the actuarial present value of the premiums equals the net present value of the insurance, for example: $\overline{A}_x - P(\overline{A}_x) i_k = 0$. Although this equation holds true at time of issuance, in future years the present value of the insurance will usually exceed the present value of the premiums. The difference between these two present values is called the benefit reserve. For example, the benefit reserve after k years for a whole life policy payable at the moment of death:

\[
k \times V(\overline{A}_x) = \overline{A}_{x+k} - P(\overline{A}_x) i_{x+k}
\]

There are many analogous formulas, but basically the calculation breaks down to:

<table>
<thead>
<tr>
<th>BENEFIT RESERVE</th>
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<tbody>
<tr>
<td>EQUALS</td>
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<tr>
<td>ACTUARIAL PRESENT VALUE OF REMAINING INSURANCE</td>
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<tr>
<td>MINUS</td>
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<tr>
<td>ACTUARIAL PRESENT VALUE OF REMAINING UNPAID PREMIUMS</td>
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</tbody>
</table>
Example 4: Find the benefit reserve after 4 years for Example 1:

\[
\ddot{a}_{x+4} = 12.13900 \\
A_{x+4} = 1 - (\frac{0.08}{1.08}) \times 12.13900 = 0.100815 \\
\bar{A}_{x+4} = 0.100815 \times (\frac{0.08}{\ln 1.08}) = 0.104796 \\
100000P(\bar{A}_x) = \pi = 654.02 \\
Reserve = 100000 V(\bar{A}_x) = (100000 \times 0.100815) - (654.02 \times 12.13900) = 2,540.35
\]

Example 5: Find the benefit reserve after 3 years for Example 2:

\[
\ddot{a}_{x+3} = 11.13855 \\
A_{x+3} = 1 - (\frac{0.08}{1.08}) \times 11.13855 = 0.174922 \\
\bar{A}_{x+3} = 0.174922 \times (\frac{0.08}{\ln 1.08}) = 0.181830 \\
200000 P^{(2)}(\bar{A}_x) = 2 \pi = 3,057.49 \\
\ddot{a}_{x+3}^{(2)} = 9.19634 \text{ (17 remaining years of payments)} \\
Reserve = 200000^{(2)} V^{(2)}(\bar{A}_x) = (200000 \times 0.181830) - (3057.49 \times 9.19634) = 8,248.18
\]

Homework

For all problems, find the following values. Insurance is payable at time of death. Use GA94 Female mortality and 7% interest.

1. The level annual premium for a whole life policy of $100,000 to an individual aged 40.
2. The level semi-annual (2 times a year) premium for a whole life policy of $200,000 to an individual age 50, where the entire cost of the insurance will be paid in 10 years.
3. The level annual premium for a 20 year term life policy of $250,000 to an individual age 35, where the entire cost of the insurance will be paid in 20 years.
4. The level quarterly (4 times a year) premium for a whole life policy of $300,000 to an individual aged 60.
5. The cash reserve of Problem 1 after 5 years.
6. The cash reserve of Problem 2 after 10 years.
7. The cash reserve of Problem 3 after 10 years.
8. The cash reserve of Problem 3 after 20 years.
9. The cash reserve of Problem 4 after 2 years.